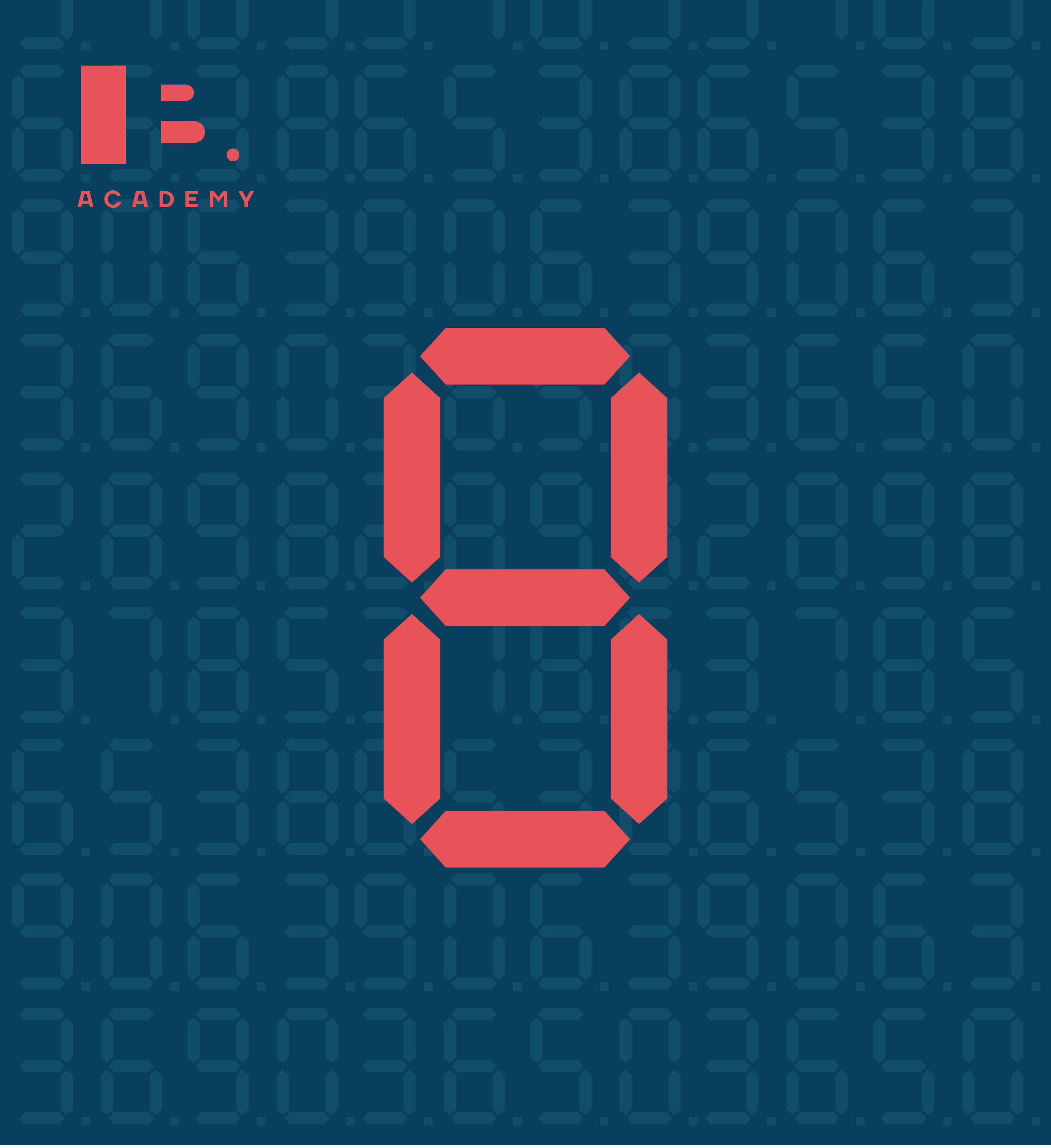

STUDY GUIDE

MATH AA HL

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ACADEMY

Mathematics Analysis and Approaches HL Study Guide

Available on www.ib.academy

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Welcome to the IB.Academy guide for Mathematics Analysis and Approaches HL.

Our Study Guides are put together by our teachers who worked tirelessly with students and schools. The idea is to compile revision material that would be easy-to-follow for IB students worldwide and for school teachers to utilise them for their classrooms. Our approach is straightforward: by adopting a step-by-step perspective, students can easily absorb dense information in a quick and efficient manner. With this format, students will be able to tackle every question swiftly and without any difficulties.

For this guide, we supplement the new topics with relevant sections from our previous Math Studies, SL and HL study resources, and with insights from our years of experience teaching these courses. We illustrate theoretical concepts by working through IB-style questions and break things down using a step-by-step approach. We also include detailed instructions on how to use the TI-Nspire™ to solve problems; most of this is also quite easily transferable to other GDC models.

The best way to apply what you have learned from the guides is with a study partner. We suggest revising with a friend or with a group in order to immediately test the information you gathered from our guides. This will help you not only process the information, but also help you formulate your answers for the exams. Practice makes better and what better way to do it than with your friends!

In order to maintain our Study Guides and to put forth the best possible material, we are in constant collaboration with students and teachers alike. To help us, we ask that you provide feedback and suggestions so that we can modify the contents to be relevant for IB studies. We appreciate any comments and hope that our Study Guides will help you with your revision or in your lessons. For more information on our material or courses, be sure to check our site at www.ib.academy.

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ALGEBRA

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Arithmetic: +/− common difference

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$
$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with $u_1 = a = 1^{\text{st}}$ term, $d =$ common difference.

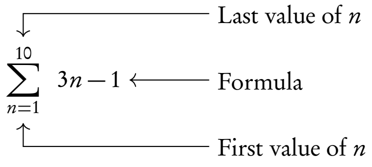
Geometric: ×/÷ common ratio

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$
$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$
$$S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}, \text{ when } -1 < r < 1$$

with $u_1 = a = 1^{\text{st}}$ term, $r =$ common ratio.

Sigma notation

A shorthand to show the sum of a number of terms in a sequence.



e.g.

$$\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot1)-1}_{n=1} + \underbrace{(3\cdot2)-1}_{n=2} + \cdots + \underbrace{(3\cdot10)-1}_{n=10} = 155$$

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Exponents

$$x^1 = x \qquad x^0 = 1$$
$$x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m\cdot n} \qquad (x\cdot y)^n = x^n \cdot y^n$$
$$x^{-1} = \frac{1}{x} \qquad x^{-n} = \frac{1}{x^n}$$
$$x^{\frac{1}{2}} = \sqrt{x} \qquad \sqrt{x}\cdot\sqrt{x} = x$$
$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}}$$

Logarithms

$$\log_a a^x = x \qquad a^{\log_a b} = b$$

Let $a^x = b$, isolate x from the exponent: $\log_a a^x = x = \log_a b$

Let $\log_a x = b$, isolate x from the logarithm: $a^{\log_a x} = x = a^b$

Laws of logarithms

I: $\log_c a + \log_c b = \log_c(a\cdot b)$

II: $\log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)$

III: $n\log_c a = \log_c(a^n)$

IV: $\log_b a = \dfrac{\log_c a}{\log_c b}$

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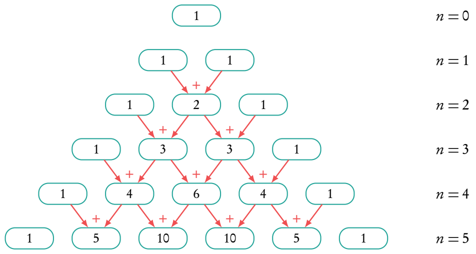
In a expansion of a binomial in the form $(a+b)^n$. Each term can be described as ${}^nC_r a^{n-r} b^r$, where ${}^nC_r$ is the coefficient.
The full expansion can be written thus

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$$

Find the coefficient using either pascals triangle



Or the nCr function on your calculator

1.1 Sequences

1.1.1 Arithmetic sequence

Arithmetic sequence the next term is the previous number + the common difference (d).

To find the common difference d, subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e. $u_{(n+1)} - u_n$.

DB 1.2 Use the following equations to calculate the n^{th} term or the sum of n terms.

$$u_n = u_1 + (n-1)d \qquad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with

$u_1 = a = 1^{\text{st}}$ term

$d =$ common difference

Often the IB requires you to first find the 1st term and/or common difference.

Finding the first term u_1 and the common difference d from other terms.

In an arithmetic sequence $u_{10} = 37$ and $u_{22} = 1$. Find the common difference and the first term.

1.	Put numbers in to n^{th} term formula	$37 = u_1 + 9d$ $1 = u_1 + 21d$
2.	Equate formulas to find d	$21d - 1 = 9d - 37$ $12d = -36$ $d = -3$
3.	Use d to find u_1	$1 - 21 \cdot (-3) = u_1$ $u_1 = 64$

1.1.2 Geometric sequence

Geometric sequence the next term is the previous number multiplied by the common ratio (r).

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e. $\frac{\text{second term } (u_2)}{\text{first term } (u_1)}$

Use the following equations to calculate the n^{th} term, the sum of n terms or the sum to infinity when $-1 < r < 1$.

DB 1.3 & 1.8

$$u_n = n^{\text{th}} \text{ term} \qquad S_n = \text{sum of } n \text{ terms} \qquad S_\infty = \text{sum to infinity}$$

$$= u_1 \cdot r^{n-1} \qquad = \frac{u_1(1-r^n)}{(1-r)} \qquad = \frac{u_1}{1-r}$$

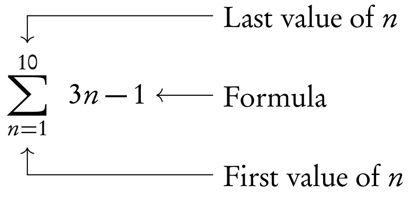
again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1st term and/or common ratio first.

1.1.3 Sigma notation

Sigma notation is a way to represent the summation of any sequence — this means that it can be used for both arithmetic or geometric series. The notation shows you the formula that generates terms of a sequence and the upper and lower limits of the terms that you want to add up in this sequence.



e.g. $\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot 1)-1}_{n=1} + \underbrace{(3\cdot 2)-1}_{n=2} + \underbrace{(3\cdot 3)-1}_{n=3} + \cdots + \underbrace{(3\cdot 10)-1}_{n=10} = 155$

Finding the first term u_1 and common ratio r from other terms.

$$\sum_{1}^{5}(\text{Geometric series}) = 3798, \sum_{1}^{\infty}(\text{Geometric series}) = 4374.$$

Find $\sum_{1}^{7}(\text{Geometric series})$

1.	Interpret the question	The sum of the first 5 terms of a geometric sequence is 3798 and the sum to infinity is 4374. Find the sum of the first 7 terms
2.	Use formula for sum of n terms	$3798 = u_1\frac{1-r^5}{1-r}$
3.	Use formula for sum to infinity	$4374 = \frac{u_1}{1-r}$
4.	Rearrange **3.** for u_1	$4374(1-r) = u_1$
5.	Substitute in to **2.**	$3798 = \frac{4374(1-r)\left(1-r^5\right)}{1-r}$
6.	Solve for r	$3798 = 4374\left(1-r^5\right)$ $\frac{3798}{4374} = 1-r^5$ $r^5 = 1-\frac{211}{243}$ $\sqrt[5]{r} = \sqrt[5]{\frac{32}{243}}$ $r = \frac{2}{3}$
7.	Use r to find u_1	$u_1 = 4374\left(1-\frac{2}{3}\right)$ $u_1 = 1458$
8.	Find sum of first 7 terms	$1458\frac{1-\left(\frac{2}{3}\right)^7}{1-\frac{2}{3}} = 4370$

1.1.4 Compound interest

Sequences can be applied to many real life situations. One of those applications is calculating the interest of a loan or a deposit. Compound interest specifically deals with interest that is applied on top of previously calculated interest. For example, if you make a deposit in a bank and reinvest the interest you will gain even more interest next time. This happens because interest is calculated not just from your initial sum, but also including your re-investments.

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

DB 1.4

Where:

FV	is the future value,
PV	is the present value,
n	is number of years,
k	is the number of compounding periods per year,
$r\%$	is the nominal annual rate of interest

Example.

A deposit of 1000$ was made in a bank with annual interest of 3% that is compounded quarterly. Calculate the balance in 5 years.

We can use our compound interest equation. Let's identify the known variables.

$$PV = 1000\$$$
$$n = 5$$
$$k = 4$$
$$r = 3\%$$
$$FV = PV \cdot \left(1 + \frac{r}{100k}\right)^{kn}$$
$$FV = 1000 \cdot \left(1 + \frac{3}{100 \cdot 4}\right)^{4 \cdot 5}$$
$$FV = 1160\$$$

1.2 Exponents and logarithms

1.2.1 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

Example.

$$x^1 = x \qquad 6^1 = 6$$

$$x^0 = 1 \qquad 7^0 = 1$$

$$x^m \cdot x^n = x^{m+n} \qquad 4^5 \cdot 4^6 = 4^{11}$$

$$\frac{x^m}{x^n} = x^{m-n} \qquad \frac{3^5}{3^4} = 3^{5-4} = 3^1 = 3$$

$$(x^m)^n = x^{m \cdot n} \qquad \left(10^5\right)^2 = 10^{10}$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad (2 \cdot 4)^3 = 2^3 \cdot 4^3 \quad \text{and} \quad (3x)^4 = 3^4 x^4$$

$$x^{-1} = \frac{1}{x} \qquad 5^{-1} = \frac{1}{5} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$

$$x^{-n} = \frac{1}{x^n} \qquad 3^{-5} = \frac{1}{3^5} = \frac{1}{243}$$

1.2.2 Fractional exponents

When doing mathematical operations (+, −, × or ÷) with fractions in the exponent you will need the following rules. These are often helpful when writing your answers in simplest terms.

Example.

$$x^{\frac{1}{2}} = \sqrt{x} \qquad 2^{\frac{1}{2}} = \sqrt{2}$$

$$\sqrt{x} \cdot \sqrt{x} = x \qquad \sqrt{3} \cdot \sqrt{3} = 3$$

$$\sqrt{xy} = \sqrt{x} \cdot \sqrt{y} \qquad \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \qquad 5^{\frac{1}{3}} = \sqrt[3]{5}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad 3^{-\frac{2}{5}} = \frac{1}{\sqrt[5]{3^2}}$$

1.2.3 Laws of logarithms

DB 1.5

Logarithms are the inverse mathematical operation of exponents, like division is the inverse mathematical operation of multiplication. The logarithm is often used to find the variable in an exponent.

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

Since $\log_a a^x = x$, so then $x = \log_a b$.
This formula shows that the variable x in the power of the exponent becomes the subject of your log equation, while the number a becomes the base of your logarithm.

Logarithms with bases of 10 and e have special notations in which their base is not explicitly noted.

$$\log_{10} x = \log x$$
$$\log_e x = \ln x$$

Remember that e is just the irrational number $2.71828\ldots$, so the same laws apply to $\ln$ as to other logarithms.

Below are the rules that you will need to use when performing calculations with logarithms and when simplifying them. The sets of equations on the left and right are the same; on the right we show the notation that the formula booklet uses while the equations on the left are easier to understand.

Laws of logarithms and change of base

DB 1.7

I:	$\log A + \log B = \log(A \cdot B)$	$\log_a(xy) = \log_a x + \log_a y$
II:	$\log A - \log B = \log\left(\frac{A}{B}\right)$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
III:	$n \log A = \log(A^n)$	$\log_a(x^m) = m \log_a x$
IV:	$\log_B A = \frac{\log A}{\log B}$	$\log_a x = \frac{\log_b x}{\log_b a}$

With the 4th rule you can change the base of a log

Next to these rules, there are a few handy things to keep in mind when working with logarithms.

$\log_a 0 = x$ is always undefined (because $a^x \neq 0$)
$x = \log_a a = 1$, which also means that $\ln e = 1$
$e^{\ln a} = a$

Solve for x in the exponent using logarithms

Solve $2^x = 13$

1.	Take the log on both sides	$\log 2^x = \log 13$
2.	Use rule **III** to take x outside	$x \log 2 = \log 13$
3.	Solve	$x = \frac{\log 13}{\log 2}$

1.3 Binomial expansion

Binomial expression an expression $(a+b)^n$ which is the sum of two terms raised to the power n.

Binomial expansion $(a+b)^n$ expanded into a sum of terms

Binomial expansions get increasingly complex as the power increases:

binomial	**binomial expansion**
$(a+b)^1$	$= a+b$
$(a+b)^2$	$= a^2+2ab+b^2$
$(a+b)^3$	$= a^3+3a^2b+3ab^2+b^3$

The general formula for each term in the expansion is ${}^nC_r a^{n-r} b^r$.

In order to find the full binomial expansion of a binomial, you have to determine the **coefficient** ${}^nC_r$ and the **powers** for each term. The powers for a and b are found as $n-r$ and r respectively, as shown by the binomial expansion formula.

Binomial expansion formula

DB 1.9

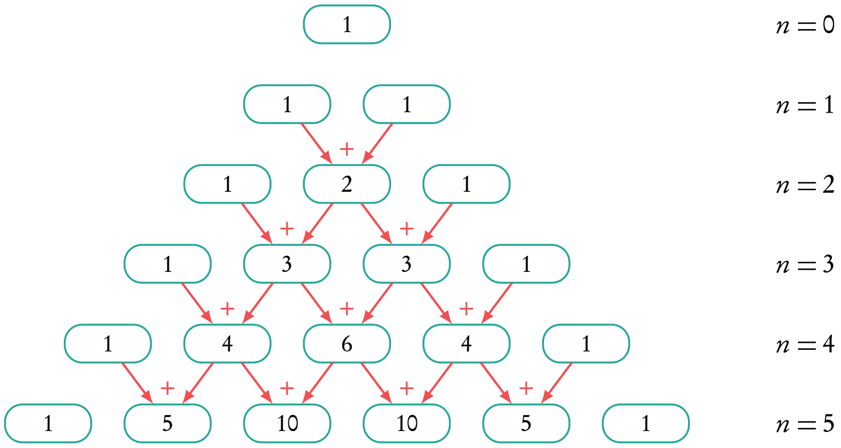
$$\begin{aligned}(a+b)^n = &\quad a^n +^n C_1 a^{n-1} b + \cdots +^n C_r a^{n-r} b^r + \cdots + b^n \\ &=^n C_0 a^n +^n C_1 a^{n-1} b +^n C_2 a^{n-2} b^2 + \ldots\end{aligned}$$

The powers decrease by 1 for a and increase by 1 for b for each subsequent term.

The sum of the powers of each term will always $= n$.

There are two ways to find the coefficients: with Pascal's triangle or the binomial coefficient function (nCr). You are expected to know both methods.

Pascal's triangle



Pascal's triangle is an easy way to find all the coefficients for your binomial expansion. It is particularly useful in cases where:

1. the power is not too high (because you have to write it out manually)
2. you need to find all the terms in a binomial expansion

Binomial coefficient functions

Combinations order is not important

$$C^n_r = \frac{n!}{(n-r)!r!} = {}^n C_r, \text{ nCr on GDC}$$

Permutations order is important

$$P^n_r = \frac{n!}{(n-r)!} = \text{number of ways of choosing } r \text{ objects out of } n = \text{nPr}$$
on GDC

Common types:

1. Arranging in a row
2. Arranging in a circle
3. Arranging letters
4. Arranging numbers

Expanding binomial expressions

Find the expansion of $\left(x-\frac{2}{x}\right)^5$

1. Use the binomial expansion formula

$a=x$, $b=-\frac{2}{x}$ and $n=5$

$$(x)^5+(5C1)(x)^4\left(-\frac{2}{x}\right)+ (5C2)(x)^3\left(-\frac{2}{x}\right)^2+(5C3)(x)^2\left(-\frac{2}{x}\right)^3+ (5C4)(x)\left(-\frac{2}{x}\right)^4+(5C5)\left(-\frac{2}{x}\right)^5$$

2. Find coefficients using Pascal's triangle for low powers or `nCr` on calculator for high powers

$$\begin{array}{lccccccccccc}
\text{Row 0:} & & & & & & 1 & & & & & \\
\text{Row 1:} & & & & & 1 & & 1 & & & & \\
\text{Row 2:} & & & & 1 & & 2 & & 1 & & & \\
\text{Row 3:} & & & 1 & & 3 & & 3 & & 1 & & \\
\text{Row 4:} & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\text{Row 5:} & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$(5C0)=1$ $(5C1)=5$ $(5C2)=10$ $(5C3)=10$ $(5C4)=5$ $(5C5)=1$

3. Put the terms and their coefficients together

$$x^5+5x^4\left(-\frac{2}{x}\right)^1+10x^3\left(-\frac{2}{x}\right)^2+ 10x^2\left(-\frac{2}{x}\right)^3+5x\left(-\frac{2}{x}\right)^4+\left(-\frac{2}{x}\right)^5$$

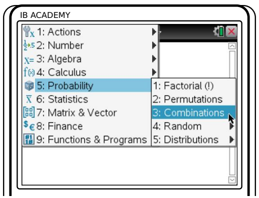
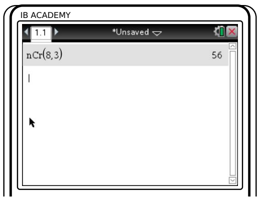
4. Simplify using laws of exponents

$$x^5-10x^3+40x-\frac{80}{x}+\frac{80}{x^3}-\frac{32}{x^5}$$

Finding a specific term in a binomial expansion

Find the coefficient of x^5 in the expansion of $(2x-5)^8$

1.	Use the binomial expansion formula	$(a+b)^n = \dots +^n C_r a^{n-r} b^r + \dots$
2.	Determine r	Since $a = 2x$, to find x^5 we need a^5. $a^5 = a^{n-r} = a^{8-r}$, so $r = 3$
3.	Plug r into the general formula	$^nC_r a^{n-r} b^r =^8 C_3 a^{8-3} b^3 =^8 C_3 a^5 b^3$
4.	Replace a and b	$^8C_3(2x)^5(-5)^3$
5.	Use `nCr` to calculate the value of the coefficient, ^nC_r	$^8C_3 = 8C3 = 56$

		
		Press menu
		`5: Probability`
		`3: Combinations`
		
		nCr(8,3) 56
		Insert the values for n and r separated by a comma

6.	Substitute and simplify	$56 \times 2^5(x^5) \times (-5)^3 = -224000(x^5)$ $\Rightarrow$ coefficient of x^5 is -224000
7.	Alternatively can be found using:	$\left(\frac{8!}{5!3!}\right) = \left(\frac{8\times7\times6}{6} = 8\times7\right)$

The IB use three different terms for these types of question which will effect the answer you should give.

Coefficient	the number before the x value
Term	the number and the x value
Constant term	the number for which there is no x value (x^0)

1.4 Induction

Unlike direct proofs, where the result follows as a logical step, mathematical induction is a form of indirect proof (the only one covered in the IB syllabus). Indirect proofs tend to require a ‘creative’ step, however through training one can recognise most forms of induction.

Proof by induction can always be split up into three components, that together prove the wanted statement:

1. → Showing the statement holds for the first case, $n = 1$
2. → Assuming the statement holds true for some value, $n = k$
3. → Proving, using the assumption in 2, that the statement holds for $n = k + 1$

Common types: $f(n) > g(n)$, $f(n) = g(n)$, $f(n) < g(n)$, Σ, $P(n)$ is a multiple / divisible.

Induction

Use induction to prove that $5 \times 7^n + 1$ is divisible by 6, $n \in \mathbb{Z}^+$

1.	Write statement in mathematical form	$P(n) = 5 \times 7^n + 1 = 6A$, $A \in \mathbb{N}$
2.	Check for $n = 1$	$P(1) = 5 \times 7^1 + 1 = 36$, which is divisible by 6
3.	Assume true for $n = k$	$5 \times \left(7^k\right) - 1 = 6A \Rightarrow 7^k = \dfrac{6A+1}{5}$
4.	Show true for $n = k + 1$	$5 \times \left(7^{(k+1)}\right) - 1 = 6B$ using assumption: $5 \times 7 \times \left(\dfrac{6A+1}{5}\right) - 1 = 6B$ $42A + 7 = 6B + 1$ $6(7A + 1) = 6B$
5.	Write concluding sentence	Hence, since $P(1)$ true and assuming $P(k)$ true, we have shown by the principle of mathematical induction that $P(k+1)$ true. Therefore, $5 \times (7^n) - 1$ is divisible by 6 for all positive integers.

Induction

Use induction to prove that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$, $n \in \mathbb{Z}^+$

1.	Check for $n=1$	$\frac{1(1+1)}{2}=\frac{2}{2}=1$ hence true for $n=1$
2.	Assume true for $n=k$	$1+2+3+\cdots+k=\frac{k(k+1)}{2}$
3.	Show true for $n=k+1$	Using assumption $\frac{k(k+1)}{2}$: $1+2+3+\cdots+k+(k+1)=$ $=\frac{k(k+1)}{2}+(k+1)$ $=\frac{k(k+1)+2(k+1)}{2}$ $=\frac{(k+2)(k+1)}{2}$ Hence true for $n=k+1$
4.	Write concluding statement	Hence, since $n=1$ is true and assuming $n=k$ true, we have shown by the principle of mathematical induction that $n=k+1$ true. Therefore, $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ is true for all positive integers.

1.5 Complex numbers

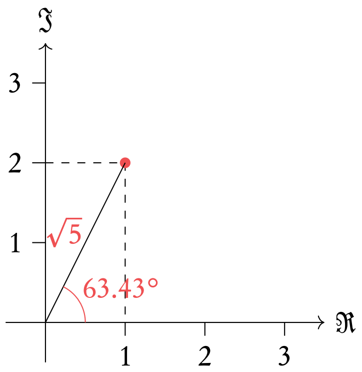
A complex number is defined as $z = a + b\mathrm{i}$. Where $a, b \in \mathbb{R}$, a is the real part ($\Re$) and b is the imaginary part ($\Im$).

$$\mathrm{i} = \sqrt{-1}$$
$$\mathrm{i}^2 = -1$$

$z = a + b\mathrm{i}$ is the Cartesian form. $z = r(\cos\vartheta + \mathrm{i}\sin\vartheta)$ is the polar form where r is the modulus and ϑ is the argument also sometimes stated as $z = r \operatorname{cis}\vartheta$.

Modulus r the absolute distance from the origin to the point.

Argument ϑ the angle between the x-axis and the line connecting the origin and the point.



Instead of working in (x, y) coordinates, polar coordinates use the distance from the origin to the point (r, modulus) and the angle between the x-axis and the modulus (argument).
$2 = 1 + 2\mathrm{i} \Rightarrow r = \sqrt{1^2 + 2^2} = \sqrt{5}$
and $\vartheta = \arctan(2) = 63.43°$
and $\sqrt{5} \times \sin(63.43) = 2$,
$\sqrt{5} \times \cos(63.43) = 1$

The conjugate of a complex number $\overline{z}$ or z^*, is defined as $z = a + b\mathrm{i} \Rightarrow \overline{z} = a - b\mathrm{i}$

1.5.1 Complex numbers in the Cartesian form

Adding and subtracting complex numbers in Cartesian form is fairly straight forward. Add real and imaginary parts to each other:

$$(2+3\mathrm{i})+(4+9\mathrm{i})=2+4+3\mathrm{i}+9\mathrm{i}=6+12\mathrm{i}$$

Multiplying complex numbers is like multiplying two parentheses:

$$\begin{aligned}(3-2\mathrm{i})(4+3\mathrm{i}) &= 3\times 4-2\mathrm{i}\times 4+3\times 3\mathrm{i}-2\mathrm{i}\times 3\mathrm{i}\\ &= 12-8\mathrm{i}+9\mathrm{i}-6\mathrm{i}^2\\ &= 12+6+\mathrm{i}\\ &= 18+\mathrm{i}\end{aligned}$$

Division, however, is slightly more complex. Conjugates play a big role here, since a complex number multiplied by its conjugate is always equal to a real number.

Rewriting of a fraction with complex numbers

Rewrite $\dfrac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in $a+b\mathrm{i}$ form.

1.	Convert the denominator into a real number by multiplying it with its conjugate.	$\dfrac{(2+6\mathrm{i})(1+2\mathrm{i})}{(1-2\mathrm{i})(1+2\mathrm{i})}$
2.	Expand the brackets and simplify, remember that $\mathrm{i}^2=-1$.	$\dfrac{2+6\mathrm{i}+4\mathrm{i}+12\mathrm{i}^2}{1-2\mathrm{i}+2\mathrm{i}-4\mathrm{i}^2}=\dfrac{-10+10\mathrm{i}}{5}$
3.	Write in $a+b\mathrm{i}$ form.	$-2+2\mathrm{i}$

1.5.2 Complex numbers in the Polar form

Polar form allows us to do some operations quicker and more efficient, such as multiplication and division of complex numbers. The formulas can be shown for the following two complex numbers $z_1 = r_1 \operatorname{cis}(\vartheta_1)$ and $z_2 = r_2 \operatorname{cis}(\vartheta_2)$. Note: $\operatorname{cis} x = \cos x + \mathrm{i} \sin x$.

Example.

Multiplication: $z_1 \times z_2 = r_1 \times r_2 \operatorname{cis}(\vartheta_1 + \vartheta_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\vartheta_1 - \vartheta_2)$

Euler's and De Moivre's theorem

These two theorems state the relationship between the trigonometric functions and the complex exponential function. This allows us to convert between Cartesian and Polar forms.

Euler's Theorem $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$

De Moivre's theorem $z^n = \left(r(\cos x + \mathrm{i} \sin x)\right)^n = r^n \left(\cos(nx) + \mathrm{i} \sin(nx)\right)$

De Moivre's theorem can be derived from Euler's through the exponential law for integer powers. $(\mathrm{e}^{\mathrm{i}x})^n = \mathrm{e}^{\mathrm{i}xn} = z^n$

De Moivre's theorem: proof by induction

Having seen the method of induction, we will now apply it to De Moivre's theorem.

Example.

Prove: $z^n = \left(\cos(x)\mathrm{i}\sin(x)\right)^n = \cos(nx) + \mathrm{i}\sin(nx)$.

1. Show true for $n = 1$

$$\left[\cos(x)\mathrm{i}\sin(x)\right]^1 = \cos(1x) + \mathrm{i}\sin(1x)$$
$$\cos(x)\mathrm{i}\sin(x) = \cos(x) + \mathrm{i}\sin(1x)$$

is true for $n = 1$

2. Assume true for $n = k$

$$\left[\cos(x) + \mathrm{i}\sin(x)\right]^k = \cos(kx) + \mathrm{i}\sin(kx)$$

3. Prove true for $n = k + 1$

$$\begin{aligned}\left[\cos(x) + \mathrm{i}\sin(x)\right]^{k+1} &= \cos\left((k+1)x\right) + \mathrm{i}\sin\left((k+1)x\right)\\ &= \left(\cos(x) + \mathrm{i}\sin(x)\right)^1\left(\cos(x) + \mathrm{i}\sin(x)\right)^k\end{aligned}$$

Inductive step: use assumption about $n = k$

$$= \left(\cos(x) + \mathrm{i}\sin(x)\right)\left(\cos(kx) + \mathrm{i}\sin(kx)\right)$$

Remember $\mathrm{i}^2 = -1$

$$= \left(\cos(x)\right)\left(\cos(kx)\right) + \left(\cos(x)\right)\left(\mathrm{i}\sin(kx)\right) + \left(\mathrm{i}\sin(x)\right)\left(\cos(kx)\right) - \left(\sin(x)\right)\left(\sin(kx)\right)$$

$$= \cos(x)\cos(kx) - \sin(x)\sin(kx) + \mathrm{i}\left(\cos(x)\sin(kx) + \sin(x)\cos(kx)\right)$$

Use of the double/half angle formulae

$$\begin{aligned}&= \cos\left(\vartheta + k\vartheta\right) + \mathrm{i}\sin\left(\vartheta + k\vartheta\right)\\ &= \cos\left((k+1)\vartheta\right) + \mathrm{i}\sin\left((k+1)\vartheta\right)\end{aligned}$$

is required result and form.

Hence, by assuming $n = k$ true, $n = k + 1$ is true. Since the statement is true for $n = 1$, it is true for all $n \in \mathbb{Z}^+$.

1.5.3 Nth roots of a complex number

n^{th} root of a complex number z is a number ω such that $\omega^n = z$.

To find n^{th} root of a complex number (in polar form, $z = r\,\text{cis}(\vartheta)$) you need to use the following formula:

$$z_{k+1} = \sqrt[n]{r} \times \text{cis}\left(\frac{\vartheta}{n} + \frac{2k\pi}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

Finding complex roots

Find 3 roots of $z^3 = 4 + 4\sqrt{3}\text{i}$ and draw them on the complex plane.

1. Rewrite the complex number in polar form.

$$r = \sqrt{4^2 + \left(4\sqrt{3}\right)^2} = \sqrt{64} = 8$$

$$\vartheta = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$z^3 = 8\,\text{cis}\left(\frac{\pi}{3}\right)$$

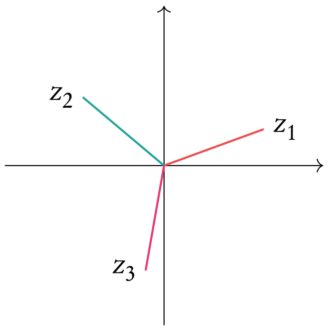
2. Insert values into formula.

$$z_1 = \sqrt[3]{8} \times \text{cis}\left(\frac{\pi}{3 \times 3}\right) = 2\,\text{cis}\left(\frac{\pi}{9}\right)$$

$$z_2 = \sqrt[3]{8} \times \text{cis}\left(\frac{\pi}{3 \times 3} + \frac{2 \times 1 \times \pi}{3}\right) = 2\,\text{cis}\left(\frac{7\pi}{9}\right)$$

$$z_3 = \sqrt[3]{8} \times \text{cis}\left(\frac{\pi}{3 \times 3} + \frac{2 \times 2 \times \pi}{3}\right) = 2\,\text{cis}\left(\frac{13\pi}{9}\right)$$

3. Draw the roots on the complex plane, they should be equally spaced out with the same length.



1.6 System of linear equations: Unique, infinite and no solutions

Solving a system of two linear equations should be familiar to most of you. There are several methods of solving it, including substitution and subtraction of equations from each other. However, sometimes there can be three equations with three unknowns or even two equations with three unknowns. It is important to identify when those equations have unique, infinite amount or no solutions at all. The easiest way to do it is to solve the simultaneous equations. It is possible to think of a system of linear equations geometrically, where the solution is at the intersection of lines or planes. Thus the intersection can be a point, a line or a plane. Here is how to identify the amount of solutions that the system of equations has:

Unique solution there is only one set of variables that satisfy all equations. Intersection is a point.

No solutions no set of variables satisfy all equations, usually you get $1 = 0$ when solving the system. No intersection of all equations in one point.

Infinite amount of solutions infinite amount of variables satisfy the equation, meaning at least one free variable. Intersection in a line or plane.

Example.

Solve the system of linear equations:

$$\begin{cases} 3x + y = 2 & (1.1) \\ 2x + 4y = -2 & (1.2) \end{cases}$$

First rewrite (1.1) as $y = 2 - 3x$ to substitute into (1.2):

$$\begin{cases} y = 2 - 3x \\ 2x + 4(2 - 3x) = -2 \end{cases}$$

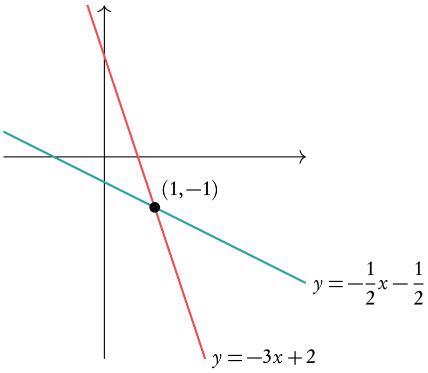
$$2x + 8 - 12x = -2$$

$$-10x = -10$$

$$x = 1$$

(1.1) $3 + y = 2 \quad \Rightarrow \quad y = -1$

The answer is: $(1, -1)$. It can also be represented graphically, as an intersection of two lines in a single point.



Example.

Solve the system of linear equations:

$$\begin{cases} x+2y+2z=4 & (1.3) \\ 4x+z=-1 & (1.4) \end{cases}$$

Rewrite equation (1.4) as $z=-1-4x$ to substitute into (1.3):

$$\begin{cases} z=-1-4x \\ x+2y+2(-1-4x)=4 \end{cases}$$

$$x+2y-2-8x=4$$

$$-7x+2y=6$$

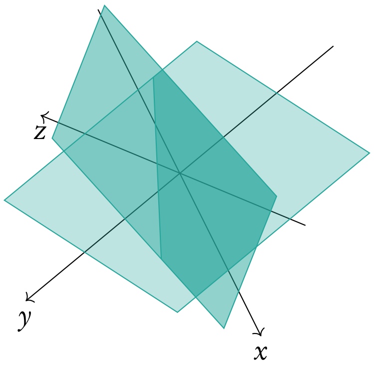
There are not enough equations to find a unique solution, so we can do a substitution.

Let $x=\lambda$.

$$\begin{aligned} -7\lambda+2y&=6 \\ 2y&=6+7\lambda \\ y&=3+3.5\lambda \\ z&=-1-4\lambda \end{aligned}$$

So our solution is: $(x,y,z)=\begin{pmatrix}0\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\3.5\\-4\end{pmatrix}$

Which means that there are infinite amount of solutions. Graphically it can be represented as two planes meeting in a line:



Example.

Solve the system of linear equations:

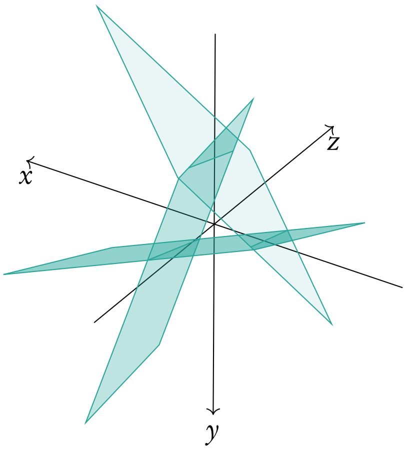
$$\begin{cases} 2x - y - z = 2 & (1.5) \\ -x + 2y - z = -1 & (1.6) \\ -x - y + 2z = 2 & (1.7) \end{cases}$$

$$(1.5) \Rightarrow \quad z = -2 + 2x - y$$

$$\begin{aligned} (1.6) \Rightarrow \quad & -x + 2y - (-2 + 2x - y) = -1 \\ & -3x + 3y = -3 \\ & x = 1 + y \end{aligned}$$

$$\begin{aligned} (1.7) \Rightarrow \quad & -(1 + y) - y + 2(-2 + 2(1 + y) - y) = 2 \\ & -1 - y - y + 2y = 2 \\ & -1 = 2 \end{aligned}$$

Which is not true, thus there are no solutions to this system of linear equations. It can be seen as three planes that do not intersect in the same point:



1.7 Partial fractions

Before you have learnt how to combine together different fractions to bring them under one denominator. However, sometimes you are required to do the opposite: split a fraction into distinct terms. In IB you will only be asked to split up fractions with two distinct linear terms in the denominator.

Solving partial fractions problems

Express $\frac{2x+1}{x^2+x-2}$ in partial fractions.

1.	Determine which linear terms make up the denominator	$x^2+x-2=(x-1)(x+2)$
2.	Equate the fraction to sum of two fractions with unknown constants as numerators and the linear terms as denominators	$\frac{2x+1}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}$
3.	Multiply by the linear terms on both sides and determine the constant terms	$2x+1=A(x+2)+B(x-1)$ $2x+1=(A+B)x+2A-B$ $2x=(A+B)x \implies A=2-B$ $1=2A-B \implies 1=2(2-B)-B$ $1=4-2B-B \implies B=1$ $A=2-1=1$
4.	Plug in constant terms into the original equation	$\frac{2x+1}{x^2+x-2}=\frac{1}{x-1}+\frac{1}{x+2}$

FUNCTIONS

2

Table of contents & cheatsheet

Definitions

Function	a mathematical relationship where each input has a single output. It is often written as $f(x)$ where x is the input
Domain	all possible x values, the input. (the domain of investigation)
Range	possible y values, the output. (the range of outcomes)
Coordinates	uniquely determines the position of a point, given by (x, y)

2.1. Types of functions 28

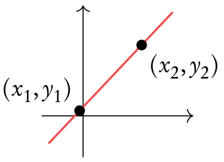
Linear functions $y = mx + c$

m is the *gradient*, c is the y *intercept*.

Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Distance: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Gradient: $m = \frac{y_2-y_1}{x_2-x_1}$

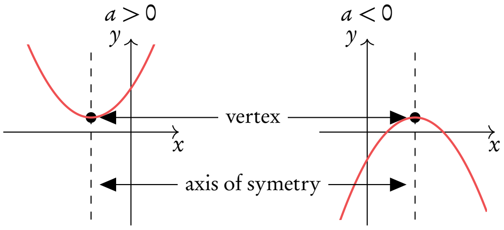


Parallel lines: $m_1 = m_2$ (same gradients)
Perpendicular lines: $m_1 m_2 = -1$

Quadratic functions $y = ax^2 + bx + c = 0$

Axis of symmetry: x-coordinate of the vertex: $x = \frac{-b}{2a}$

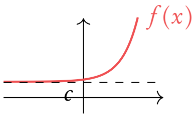
Factorized form: $y = (x+p)(x+q)$



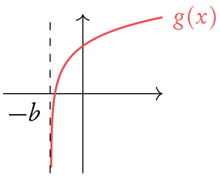
If $a = 1$ use the factorization method $(x+p)\cdot(x+q)$
If $a \neq 1$ use the quadratic formula
When asked excplicity complete the square

Vertex form: $y = a(x-h)^2 + k$
Vertex: (h, k)

Exponential
$f(x) = a^x + c$

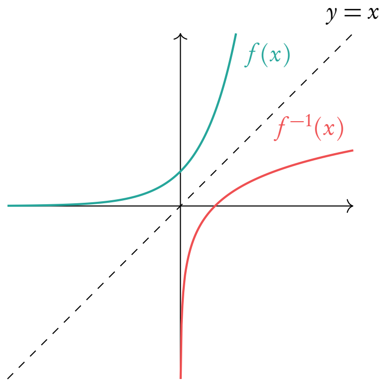


Logarithmic
$g(x) = \log_a(x+b)$



2.2. Rearranging functions 38

Inverse function, $f^{-1}(x)$ reflection of $f(x)$ in $y = x$.



Composite function, $(f \circ g)(x)$ is the combined function f of g of x.

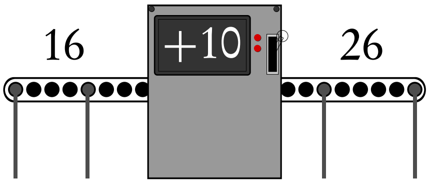
When $f(x)$ and $g(x)$ are given, replace x in $f(x)$ by $g(x)$.

Transforming functions

Change to $f(x)$	Effect
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $\frac{1}{a}$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis

2.1 Types of functions

Functions are mathematical relationships where each input has a single output. You have probably been doing functions since you began learning maths, but they may have looked like this:



Algebraically this is:
$f(x) = x + 10$,
here $x = 16$, $y = 26$.

We can use graphs to show multiple outputs of y for inputs x, and therefore visualize the relation between the two. Two common types of functions are linear functions and quadratic functions.

2.1.1 Linear functions

Linear functions $y = mx + c$ increases/decreases at a constant rate m, where m is the gradient and c is the y-intercept

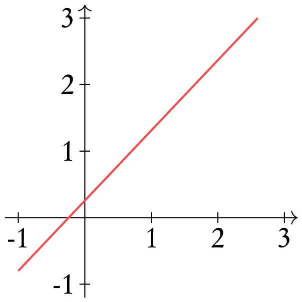
Midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Distance $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

Parallel lines $m_1 = m_2$ (equal gradients)

Perpendicular lines $m_1 m_2 = -1$



Straight line equations are sometimes written in two other forms, which you should be comfortable rearranging them to and from:

$ax + by + d = 0$	general form
$y - y_1 = m(x - x_1)$	point-slope form

Example.

Determine the midpoint, length and gradient of the straight line connecting the two points $P_1(2, 8)$ and $P_2(6, 3)$

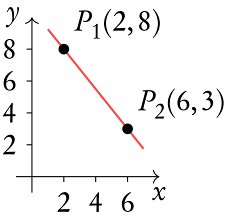
Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+6}{2}, \frac{8+3}{2}\right) = (4, 5.5)$

Distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(6-2)^2 + (3-8)^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{41}$

Gradient: $m = \frac{y_2 - y_1}{x_2 - x_1} = m = \frac{3-8}{6-2} = -\frac{5}{4}$

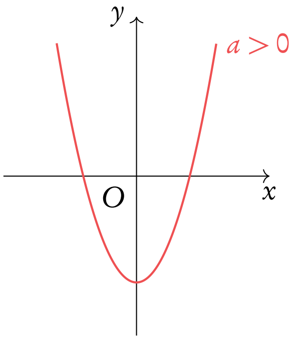
Parallel line: $-\frac{5}{4}x + 3$

Perpendicular line: $\frac{4}{5}x + 7$

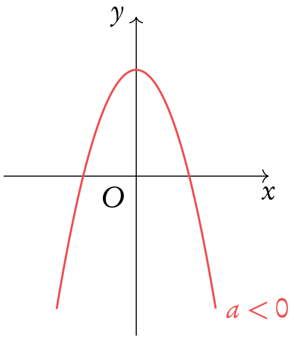


2.1.2 Quadratic functions

Graphed, quadratic functions make a parabolic shape; they increase/decrease at an increasing rate.



$a > 0$, positive quadratic



$a < 0$, negative quadratic

Quadratic functions $y = ax^2 + bx + c = 0$

Axis of symmetry $x = \frac{-b}{2a}$ = x-coordinate of vertex

DB 2.6

Roots the x-intercept(s). Algebraically, roots can be found through factorisation or using the quadratic formula

Quadratic formula used to find the roots if $a \neq 1$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Discriminant Δ the $b^2 - 4ac$ part of the formula, which can be used to determine how many x-intercepts a quadratic equation has

$$\Delta > 0 \quad \Rightarrow 2 \text{ roots}$$
$$\Delta = 0 \quad \Rightarrow 1 \text{ root}$$
$$\Delta < 0 \quad \Rightarrow \text{no real roots}$$

Roots of quadratic equations are also referred to as ‘solutions’ or ‘zeros’.

Two more standard ways to write quadratic equations are:

$y = a(x - h)^2 + k$ with the vertex at (h, k)

$y = a(x - p)(x - q)$ with x-intercepts $(p, 0)(q, 0)$

Solving quadratic equations by factorisation

Solve $x^2-5x+6=0$

1.	Set up a system of equations $p+q=b$ and $p\times q=c$	$\left.\begin{matrix} p+q=-5 \\ p\times q=6 \end{matrix}\right\} p=-2 \quad \text{and} \quad q=-3$
2.	Plug the values for p and q into $(x+p)(x+q)$	$(x-2)(x-3)=x^2-5x+6$
3.	Equate each part to 0 $(x+p)=0$, $(x+q)=0$, and solve for x	$\left.\begin{matrix} (x-2)=0 \\ (x-3)=0 \end{matrix}\right\} x=2 \quad \text{or} \quad x=3$

Solving quadratic equations using the quadratic formula

Solve $3x^2-8x+4=0$

1.	Calculate the discriminant $\Delta=b^2-4ac$	$\Delta=(-8)^2-4\cdot 3\cdot 4=16$
2.	Use the discriminant to determine the number of solutions	$\Delta>0$ so 2 solutions
3.	Find solutions using quadratic formula $x=\dfrac{-b\pm\sqrt{\Delta}}{2a}$	$x=\dfrac{8\pm\sqrt{16}}{2\cdot 3}=\dfrac{8\pm 4}{6}$ $\left.\begin{matrix} =\dfrac{8-4}{6}=\dfrac{4}{6} \\ =\dfrac{8+4}{6}=2 \end{matrix}\right\} \Rightarrow \quad x=\dfrac{2}{3} \quad \text{or} \quad x=2$

Through a method called completing the square, you can rearrange a quadratic function into the form $y = a(x-h)^2 + k$. This way you can find the coordinates of the vertex (the minimum or maximum). For the exam you will always be asked explicitly to use this method.

Find the vertex by completing the square

Express $f(x) = 4x^2 - 2x - 5 = 0$ in the form $y = a(x-h)^2 + k$. Hence, find the coordinate of the vertex of $f(x)$.

1.	Move c to the other side	$4x^2 - 2x = 5$
2.	Divide by a	$x^2 - \frac{1}{2}x = \frac{5}{4}$
3.	Calculate $\left(\frac{x \text{ coeficient}}{2}\right)^2$	$\left(\frac{-\frac{1}{2}}{2}\right)^2 = \frac{1}{16}$
4.	Add this term to both sides	$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{5}{4} + \frac{1}{16}$
5.	Factor perfect square and bring constant back	$\left(x - \frac{1}{4}\right)^2 - \frac{21}{16} = 0$ $\Rightarrow$ minimum point=$\left(\frac{1}{4}, -\frac{21}{16}\right)$

2.1.3 Functions with asymptotes

Asymptote a straight line that a curve approaches, but never touches.

A single function can have multiple asymptotes: horizontal, vertical and in rare cases diagonal. Functions that contain the variable (x) in the denominator of a fraction and exponential and logarithmic functions will always have asymptotes.

Vertical asymptotes

Vertical asymptotes occur when the denominator is zero, as dividing by zero is undefinable. Therefore if the denominator contains x and there is a value for x for which the denominator will be 0, we get a vertical asymptote.

Example.

In the function $f(x) = \frac{x}{x-4}$ the denominator is 0 when $x = 4$, so this line forms the a vertical asymptote.

Horizontal asymptotes

Horizontal asymptotes are the value that a function tends to as x becomes really big or really small; technically speaking to the limit of infinity, $x \to \infty$. The general idea is then that when x is large, other parts of the function not involving x become insignificant and so can be ignored.

Example.

In the function $f(x) = \frac{x}{x-4}$, when x is small the 4 is important.

$$x = 10 \qquad 10 - 4 = 6$$

But as x gets bigger the 4 becomes increasingly insignificant

$$x = 100 \qquad 100 - 4 = 96$$

$$x = 10000 \qquad 10000 - 4 = 9996$$

Therefore as we approach the limits we can ignore the 4.

$$\lim_{x \to \infty} f(x) = \frac{x}{x} = 1$$

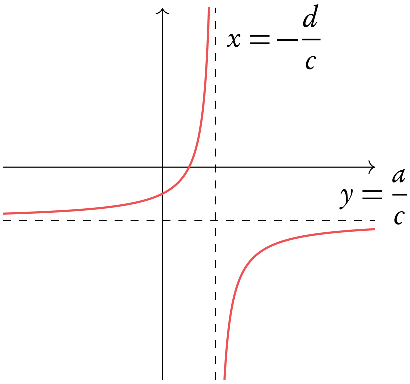
So there is a horizontal asymptote at $y = 1$.

2.1.4 Special Functions

The function $\dfrac{ax+b}{cx+d}$

Rational function of the form $y = \dfrac{ax+b}{cx+d}$

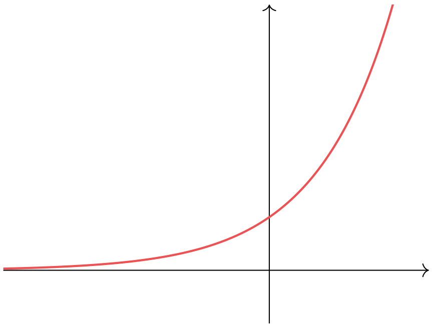
The graph made by this function has one horizontal and one vertical asymptote. The graph is not continuous in all points, but splits into two parts. Both of the parts approach horizontal asymptote at either negative or positive large values of x and approach horizontal asymptote at either negative or positive large values of y. Also both parts are located in different "corners" of the coordinate system.
The vertical asymptote occurs where denominator is equal to 0, meaning $x = -\dfrac{d}{c}$. The graph is not defined at that point.
The horizontal asymptote occurs at very large values of x, meaning that asymptote occurs at $y = \lim_{x\to\infty} \dfrac{ax+b}{cx+d} = \dfrac{a}{c}$.



The function a^x and its graph

Exponential function: $y = a^{(x+b)} + c$

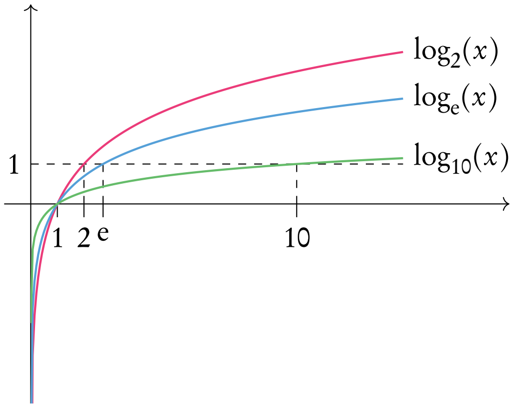
Graph is upward-sloping with bigger x values. It has asymptote at $y = c$, when x approaches large negative values.
If $c \geq 0$, then the graph never becomes negative, also meaning that there are no roots.
The graph crosses y-axis at $\left(0, a^b + c\right)$, meaning that if $b = 0$ and $c = 0$, then that point is $(0, 1)$.
Solving the exponential equations often requires logarithms.



The function $\log(x)$ and its graph

Logarithmic function: $y = \log_a(x)$

Logarithmic function is inverse to exponential function, thus the graph has the same shape, reflected along $y = x$ line. It is down-sloping with bigger x values and has an asymptote at $x = 0$.
The graph always has a single root at $x = 1$.



The function $\dfrac{ax^2+bx+c}{dx+e}$

Rational function of the form $\dfrac{ax^2+bx+c}{dx+e}$

The function looks very interesting, because there is not only a vertical asymptote but also seemingly an asymptote that is neither vertical or horizontal. That is an "oblique asymptote". It occurs when the degree of the numerator is bigger than the degree of the denominator. To find an equation of the oblique asymptote, one needs to do polynomial division (explained a few pages later).

Example.

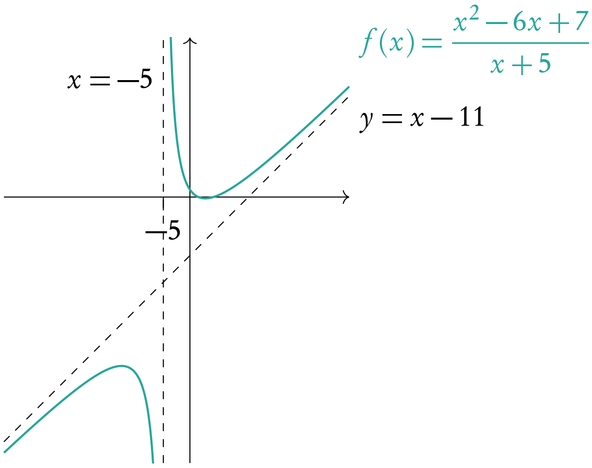
Find asymptotes of $f(x)=\frac{x^2-6x+7}{x+5}$

By doing polynomial long division one can find that it equals to:

$$\frac{x^2-6x+7}{x+5}=x-11+\frac{48}{x+5}$$

So the oblique asymptote is $y=x-11$.
The second (vertical) asymptote occurs when denominator is equal to 0:

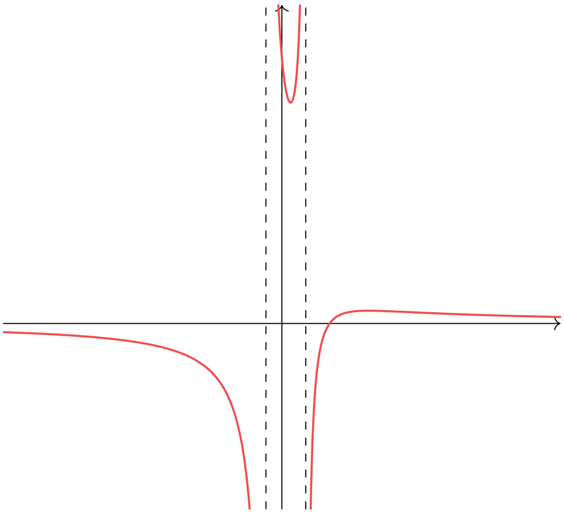
$$x+5=0$$
$$x=-5$$



The function $\dfrac{ax+b}{cx^2+dx+e}$

Rational function of the form $\dfrac{ax+b}{cx^2+dx+e}$

The degree of the denominator is bigger than the degree of the numerator, thus there are no oblique asymptotes. This also means that towards $x=\pm\infty$ the function approaches value $y=0$. So horizontal asymptote is simple: $y=0$. To find vertical ones we need to find when the denominator is equal to 0: $cx^2+dx+e=0$. This is a quadratic equation and can have up to two solutions. The amount of solutions determine the amount of vertical asymptotes.



2.1.5 Describing functions

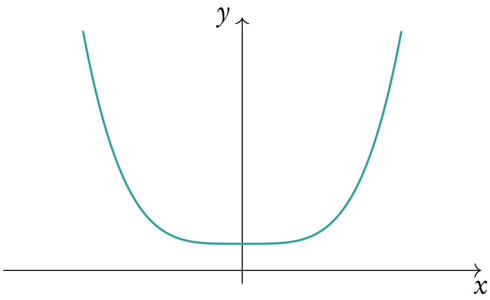
Even and odd functions

When $f(x)=f(-x)$ we describe the function as *even* or a graph symmetrical over the y-axis.

Example.

An even function: $f(x)=x^4+2$

Testing algebraically substitute $(-x)$: $f(-x)=-x^4+2=x^4+2$



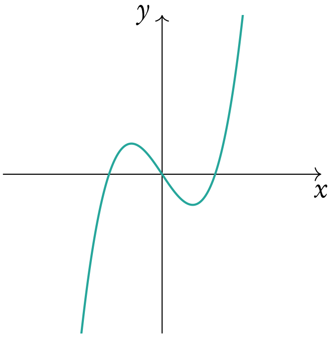
The graph is symmetrical over the y-axis.

When $-f(x)=f(-x)$ we describe the function as *odd* or a graph has rotational symmetry with respect to the origin.

Example.

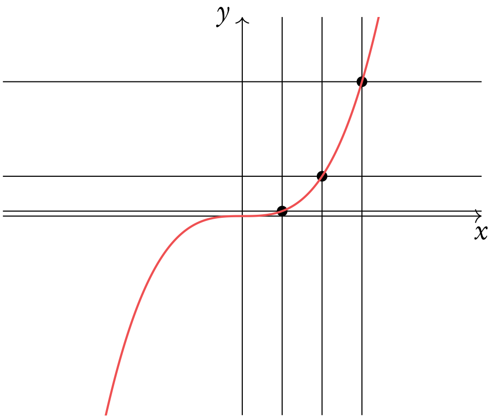
An odd function: $f(x)=x^3-x$

Testing algebraically. $-f(x)=-x^3+x$.
Substitute $(-x)$: $f(-x)=-x^3+x=-x^3+x=-f(x)$



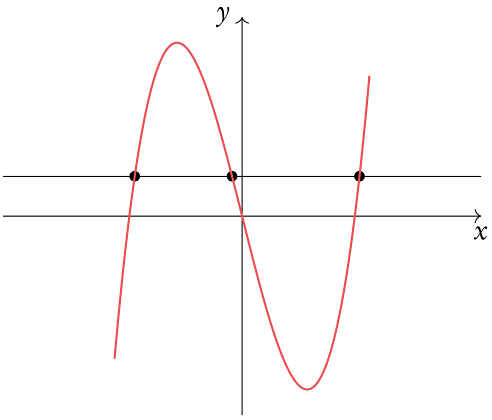
The graph has a rotational symmetry with respect to the origin.

One to one function



A one to one function is a function for which every element of the range of the function correspond to exactly one element of the domain.
Can be tested with horizontal and vertical line test.

Many to one functions

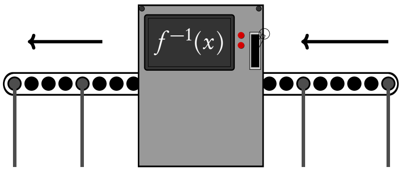


A many-to-one function is a defined as a function where there are y-values that have more than one x-value mapped onto them.

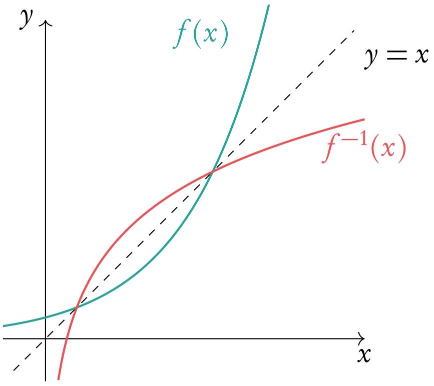
2.2 Rearranging functions

2.2.1 Inverse functions, $f^{-1}(x)$

Inverse functions are the reverse of a function. Finding the input x for the output y. You can think of it as going backwards through the number machine.



This is the same as reflecting a graph in the $y = x$ axis.



Finding the inverse function.

$f(x) = 2x^3 + 3$, find $f^{-1}(x)$

1.	Replace $f(x)$ with y	$y = 2x^3 + 3$
2.	Solve for x	$y - 3 = 2x^3$ $\Rightarrow \dfrac{y-3}{2} = x^3$ $\Rightarrow \sqrt[3]{\dfrac{y-3}{2}} = x$
3.	Replace x with $f^{-1}(x)$ and y with x	$\sqrt[3]{\dfrac{x-3}{2}} = f^{-1}(x)$

2.2.2 Composite functions

Composite functions are a combination of two functions.

$$(f \circ g)(x) \qquad \text{means } f \text{ of } g \text{ of } x$$

To find the composite function above substitute the function of $g(x)$ into the x of $f(x)$.

Remember
$f \circ g(x) \neq g \circ f(x)$

Example.

Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x)$: replace x in the $f(x)$ function with the entire $g(x)$ function

$$(2g(x)) + 3 = 2x^2 + 3$$

$(g \circ f)(x)$: replace x in the $g(x)$ function with the entire $f(x)$ function

$$(f(x))^2 = (2x+3)^2$$

2.2.3 Transforming functions

By adding and/or multiplying by constants we can transform a function into another function.

Change to $f(x)$	Effect
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $1/a$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left

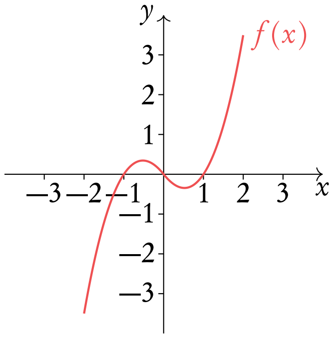
Exam hint: describe the transformation with words as well to guarantee marks.

Always do translations last

Transforming functions $f(x) \rightarrow af(x+b)$

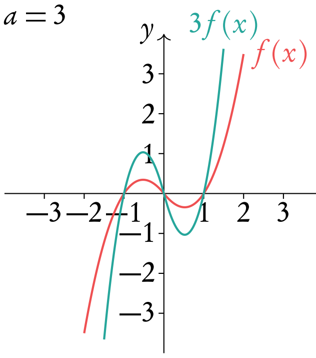
Given $f(x)=\frac{1}{4}x^3+x^2-\frac{5}{4}x$, draw $3f(x-1)$.

1. Sketch $f(x)$



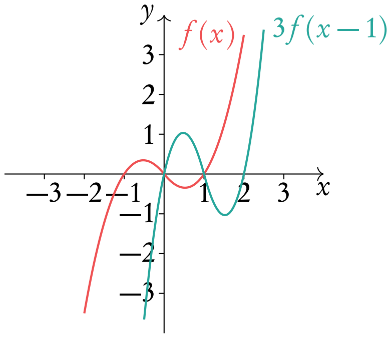
2. Stretch the graph by the factor of a

$a=3$



3. Move graph by $-b$

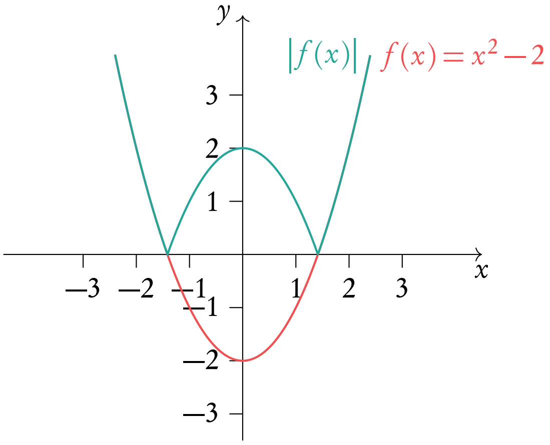
Move graph by 1 to the *right*



Example.

Absolute value: $|f|$

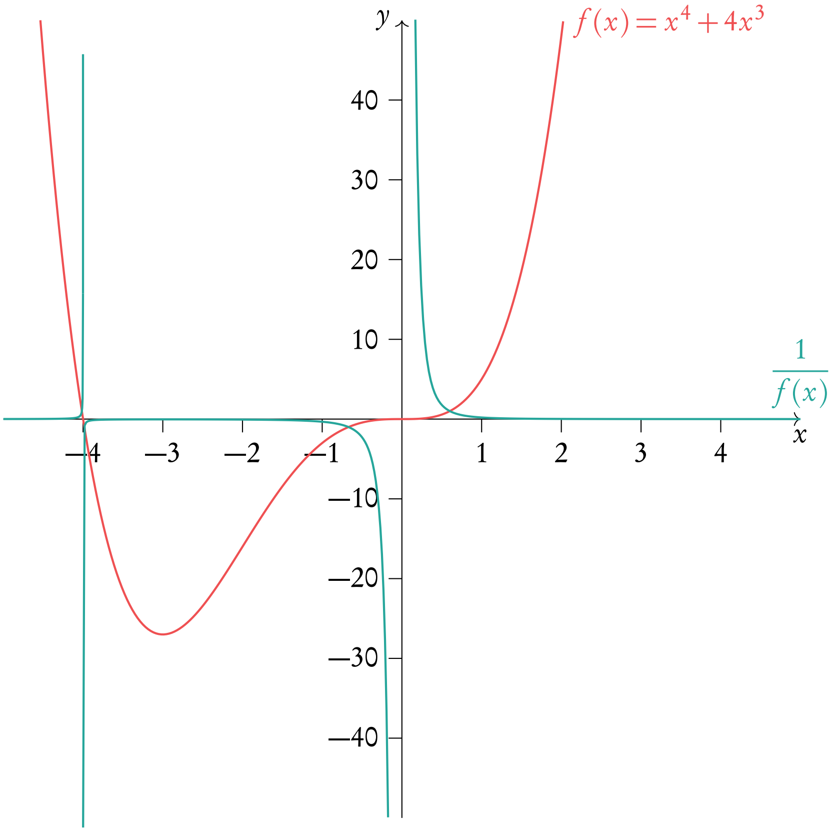
$f(x) = x^2 - 2 \Rightarrow |f(x)| = ?.$



Example.

Reciprocal: $\frac{1}{f(x)}$

$f(x) = x^4 + 4x^3$ so: $\frac{1}{f(x)} = \frac{1}{x^4 + 4x^3}$

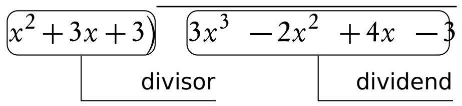


2.2.4 Polynomial long division

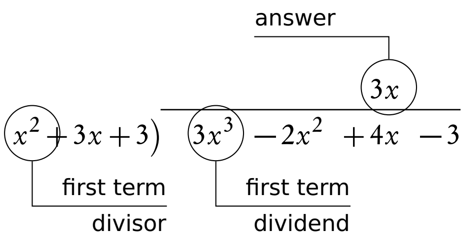
When we need to divide one polynomial by another we use *polynomial long division*. The number to be divided is called the ‘dividend’. The number which divides it is called ‘divisor’.

Polynomial long division

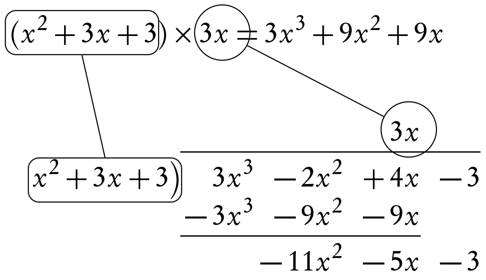
Divide $3x^3-2x^2+4x-3$ by x^2+3x+3:



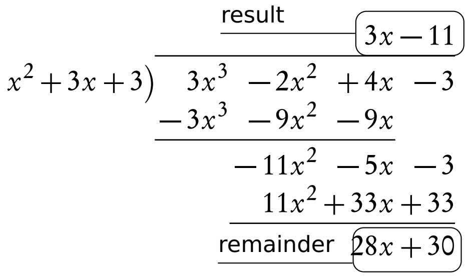
1. Divide the first term of the dividend by the first term of the divisor



2. Multiply the divisor by this answer and subtract this from our dividend



3. Divide the result of the substituition by the first term of the divisor. Repeat the process until this is no longer possibile



4. Write the answer:

$$\text{result} + \frac{\text{remainder}}{\text{divisor}}$$

$$3x-11+\frac{28x+30}{x^2+3x+3}$$

2.3 The factor and remainder theorem

Remainder theorem when we divide a polynomial $f(x)$ by $x-c$ the remainder r equals $f(c)$

Let's say

$$f(x) \div (x-c) = q(x) + r$$

where r is the remainder. We also know

$$f(x) = (x-c)q(x) + r$$

If we now substitute x with c

$$f(c) = (c-c)q(c) + r$$

but $c - c = 0$, therefore

$$f(c) = r$$

Factor theorem when $f(c) = 0$ then $x-c$ is a factor of the polynomial

2.4 Fundamental theorem of algebra

The fundamental theorem of algebra any polynomial of degree n has n roots.

A degree of a polynomial is the largest exponent.

Example.

If $f(x) = 4x^3 + 3x^2 + 7x + 9$ then it is a polynomial at degree 3, and according to the fundamental theorem of algebra, will have 3 roots.

Any polynomial can be rewritten/factorized to include the roots:

$$a(x-r_1)(x-r_2)(x-r_3)\cdots$$

where $r_1, r_2, r_3, \ldots,$ are all roots.

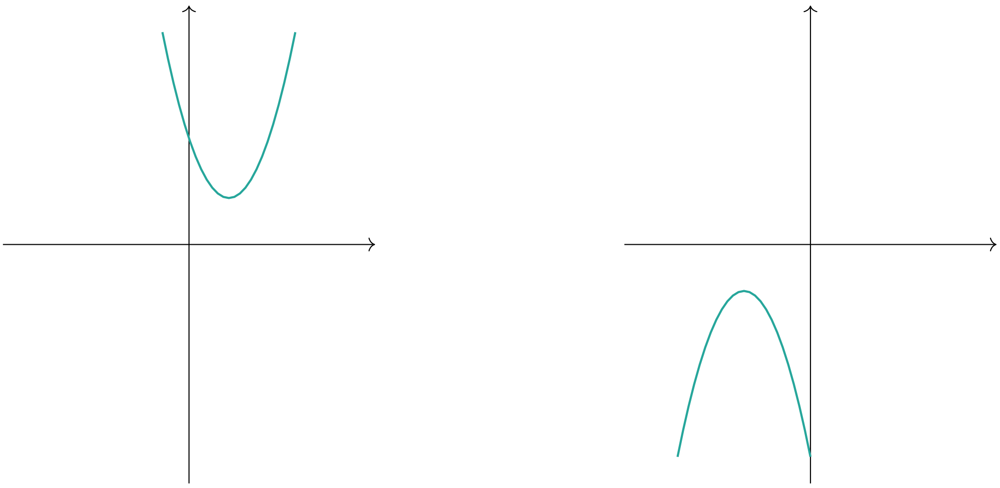
Note: some polynomials will have "double" or "triple" roots. Some may also have complex roots. Therefore a polynomial of degree 4 can have 4 real roots (of which 2,3 or 4 could be the same) or 4 complex roots (of which 2,3 or 4 could be the same) or 2 real and 2 complex roots.

Below are several examples of such situations.

Example.

Complex roots of quadratic equations always come in conjugate pairs.

For example: $2x^2-3x+4$ has complex roots, i.e. $(b^2-4ac)<0$. The roots are $x=\frac{3}{4}+\frac{i\sqrt{7}}{4}$ and $x=\frac{3}{4}-\frac{i\sqrt{7}}{4}$ can be shown graphically.



Example.

$f(x)=x^2$ is a polynomial of degree 2, so it has 2 roots. However, the only root is $x=0$. It means that $x=0$ is a double root, meaning that the graph has a local minimum (or maximum) at that point.

Thus sometimes the polynomial can have a factor of $\left(x-r_c\right)^d$, where r_c is a root and d is a number of occurrences of that root.

Example.

Rewrite $f(x)$ in a factorised form with real coefficients, where $f(x)=x^5+x^4-8x^3+ax^2+bx+24$ with real coefficients a and b. It is also known that $f(x)$ has a root $x=-1+\mathrm{i}$ and one local minimum at $x>0$.

First, it is important to remember that all roots come in conjugate pairs, meaning that $f(x)$ also has a root $x=-1-\mathrm{i}$. From there it is possible to figure out that:

$$x+1=\pm\mathrm{i}$$

$$(x+1)^2=-1$$

$$x^2+2x+2=0$$

Which is one of the factors of $f(x)$. There are two ways to proceed, first one involves polynomial division, another involves sum and product of roots.

1. Since we know one factor of $f(x)$, we can perform polynomial division to find other roots.

$$
\begin{array}{rrrrrrr}
 & & & x^3 & -x^2 & -8x & +(18+1a) \\ \cline{2-7}
x^2+2x+2\big) & x^5 & +x^4 & -8x^3 & +ax^2 & +bx & +24 \\
 & -x^5 & -2x^4 & -2x^3 & & & \\ \cline{2-4}
 & & -x^4 & -10x^3 & +ax^2 & & \\
 & & x^4 & +2x^3 & +2x^2 & & \\ \cline{3-5}
 & & & -8x^3 & +(2+1a)x^2 & +bx & \\
 & & & 8x^3 & +16x^2 & +16x & \\ \cline{4-6}
 & & & & (18+1a)x^2 & +\left(16+1b\right)x & +24 \\
 & & & & -(18+1a)x^2 & -2(18+1a)x & -2(18+1a) \\ \cline{5-7}
 & & & & & \multicolumn{2}{r}{\left(34+1+-2a+1b\right)x+(42+1+-2a)}
\end{array}
$$

Since we know that we divided by one factor of the function, the remainder has to be equal to zero. Thus:

$$
\begin{aligned}
-2a-12&=0\\
a&=-6\\
b+16-2a-36&=0\\
b+16+12-36&=0\\
b&=8
\end{aligned}
$$

Now we know values for a and b, so we can use them in what is left of our polynomial:

$$x^3-x^2-8x+(a+18)=x^3-x^2-8x+12$$

So we just need to solve our cubic equation. One way is to try to plug in specific values. For cubic polynomial $g(x)=ax^3+bx^2+cx+d$, one of the roots is usually some factor of $\frac{d}{a}$, so that it would be a rational fraction. As an example, our possible roots here are: ±12, ±6, ±4, ±3, ±2, ±1. By trying different values, you can find that $x = 2$ is one of our roots. Then it is required to perform polynomial division once again to get:

$$\left(x^3 - x^2 - 8x + 12\right) \div (x - 2) = x^2 + x - 6 = (x + 3)(x - 2)$$

As you can see, $x = 2$ is a double root, since it comes up again. Thus our full factorised $f(x)$ looks like this:

$$f(x) = \left(x^2 + 2x + 2\right)(x - 2)^2(x + 3)$$

2. Another method requires formulas for sum and product of roots. Since we know that we have a double root (because of the minimum), we can easily find both roots. Let's assume that double root is α and the other root is β.

   Sum of roots:

$$-1 + \mathrm{i} - 1 - \mathrm{i} + \beta + \alpha + \alpha = -\frac{1}{1}$$
$$2\alpha + \beta = 1$$

   Product of roots:

$$(-1 + \mathrm{i})(-1 - \mathrm{i}) \times \beta \times \alpha \times \alpha = (-1)^5 \times \frac{24}{1}$$
$$2 \times \beta \times \alpha \times \alpha = -24$$
$$\beta \times \alpha^2 = -12$$

   Now solve simultaneous equations:

$$\beta = \frac{-12}{\alpha^2}$$
$$2\alpha + \frac{-12}{\alpha^2} = 1$$
$$2\alpha^3 - \alpha^2 - 12 = 0$$

   Again, solve the cubic equation, to get $\alpha = 2$ and thus $\beta = 3$. Now factorise to get the same answer:

$$f(x) = \left(x^2 + 2x + 2\right)(x - 2)^2(x + 3)$$

2.5 Sums and products of roots

For any polynomial of form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$:

Sum of roots: $-\dfrac{a_{n-1}}{a_n}$

Product of roots: $(-1)^n \times \dfrac{a_0}{a_n}$

Example.

Given $x^2 + 8x + k = 0$, find both roots, when one is 3 times larger than another. Then find value of k.

First we need to find out what both roots are. Let $x_1 = \alpha$, then $x_2 = 3\alpha$. Using formula we get:

$$\text{Sum of roots: } \frac{-8}{1} \Rightarrow \text{Thus: } x_1 + x_2 = \frac{-8}{1}$$

$$\alpha + 3\alpha = -8$$

$$\alpha = -2 = x_1$$

$$x_2 = 3 \times -2 = -6$$

It means that we can factorise original polynomial as $(x+2)(x+6) = 0$, giving:

$$(x+2)(x+6) = x^2 + 8x + 12$$

Therefore, $k = 12$.

VECTORS

3

Table of contents & cheatsheet

Definitions

Vector	a geometric object with *magnitude* (length) and *direction*, represented by an *arrow*.
Collinear points	points that lie on the same line
Unit vector	vector with magnitude 1
Base vector	$\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$

3.1. Working with vectors 48

Vector from point O to point A: $\vec{OA}=\vec{a}=\begin{pmatrix}3\\2\end{pmatrix}$

Vector from point O to point B: $\vec{OB}=\vec{b}=\begin{pmatrix}-1\\1\end{pmatrix}$

Can be written in two ways:

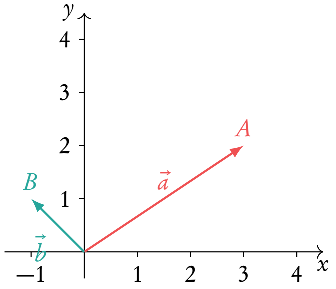
$$\vec{a}=\begin{pmatrix}3\\2\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$$\vec{a}=3i+2j+0k=3i+2j$$

Length of $\vec{a}$: $|\vec{a}|=\sqrt{x^2+y^2}=\sqrt{3^2+2^2}=\sqrt{13}$

Addition & multiplication: $\vec{a}+2\vec{b}=\begin{pmatrix}3\\2\end{pmatrix}+2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}-2\\2\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$

Subtraction: $\vec{a}-\vec{b}=\begin{pmatrix}3\\2\end{pmatrix}-\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$



3.2. Equations of lines 50

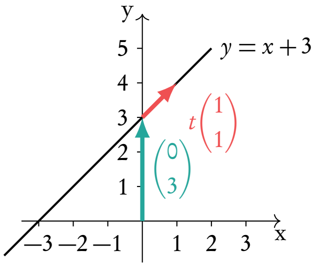
Example of a line:

$$r=\begin{pmatrix}0\\3\end{pmatrix}+t\begin{pmatrix}1\\1\end{pmatrix}$$

direction vector

parameter

position vector



3.3. Dot product 52

The dot product of two vectors $\vec{c}\cdot\vec{d}$ can be used to find the angle between them.

Let $\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}, \vec{d}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$:

$$\vec{c}\cdot\vec{d}=|\vec{c}||\vec{d}|\cos\vartheta$$

$$\vec{c}\cdot\vec{d}=c_1d_1+c_2d_2+c_3d_3$$

ACADEMY

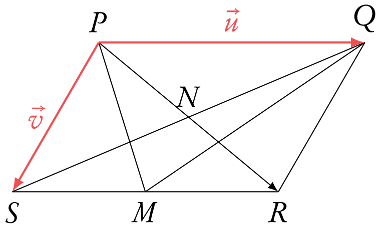
3.1 Working with vectors

Vectors are a geometric object with a *magnitude* (length) and *direction*. They are represented by an *arrow*, where the arrow shows the direction and the length represents the magnitude.

So looking at the diagram we can see that vector $\vec{u}$ has a greater magnitude than $\vec{v}$. Vectors can also be described in terms of the points they pass between. So

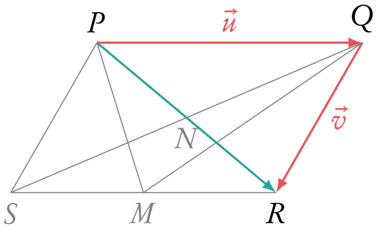
$$\begin{cases}\vec{u}=\vec{PQ}\\ \vec{v}=\vec{PS}\end{cases}$$

with the arrow over the top showing the direction.

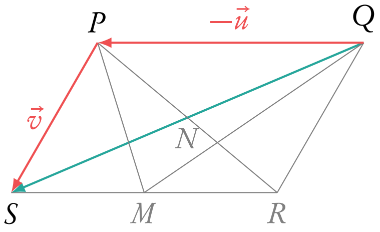


You can use vectors as a geometric algebra, expressing other vectors in terms of $\vec{u}$ and $\vec{v}$. For example

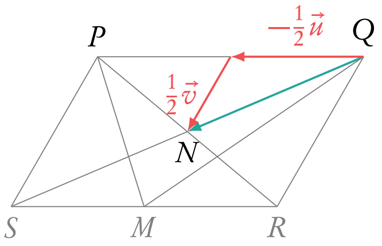
$$\vec{PR}=\vec{u}+\vec{v}$$



$$\vec{QS}=-\vec{u}+\vec{v}$$



$$\vec{QN}=\frac{1}{2}\left(-\vec{u}+\vec{v}\right)$$



This may seem slightly counter-intuitive at first. But if we add in some possible figures you can see how it works. If $\vec{u}$ moves 5 units to the left and $\vec{v}$ moves 1 unit to the right (—left) and 3 units down.

Then $\vec{PR}=\vec{u}+\vec{v}=5$ units to the left —1 unit to the right and 3 units down = 4 units to the left and 3 units down.

3.1.1 Vectors with value

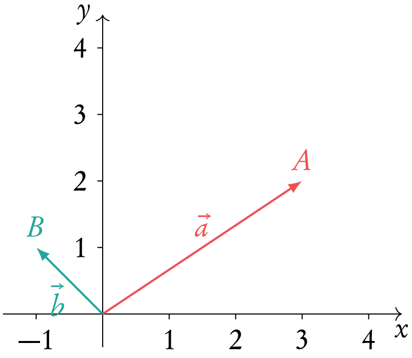
Formally the value of a vector is defined by its direction and magnitude within a 2D or 3D space. You can think of this as the steps it has to take to go from its starting point to its end, moving only in the x, y and z axis.

Vector from point O to point A:

$$\vec{OA} = \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Vector from point O to point B:

$$\vec{OB} = \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Vectors can be written in two ways:

Note: unless told otherwise, answer questions in the form used in the question.

1. $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where the top value is movement in the x-axis. Then the next is movement in the y and finally in the z. Here the vector is in 2D space as there is no value for the z-axis.
2. as the sum of the three base vectors:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\vec{i}$ is moving 1 unit in the x-axis, $\vec{j}$ 1 unit in the y-axis and $\vec{k}$ 1 unit in the z-axis.

$$\vec{a} = 3i + 2j + 0k = 3i + 2j$$

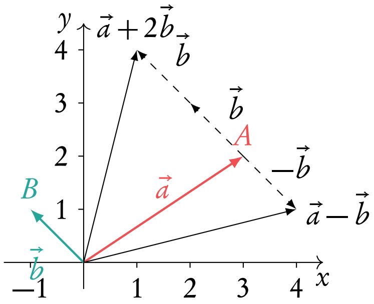
When we work with vectors we carry out the mathematical operation in each axis separately. So x-values with x-values and so on.

Addition & multiplication:

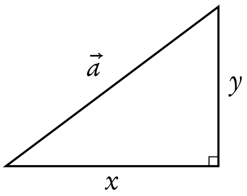
$$\vec{a}+2\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}+2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}+\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Subtraction:

$$\vec{a} - \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



However it must be remembered that vector notation does not give us the actual length (magnitude) of the vector. To find this we use something familiar.



Length of $\vec{a}$:

$$\begin{aligned}|\vec{a}| &= \sqrt{x^2+y^2}\\ &= \sqrt{3^2+2^2} = \sqrt{13}\end{aligned}$$

Sometimes you will be asked to work with unit vectors. These are vectors with a magnitude of 1. We can convert all vectors to unit vectors.

Determine the unit vector $\hat{a}$ in the direction of any vector $\vec{a}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}$$

3.2 Equations of lines

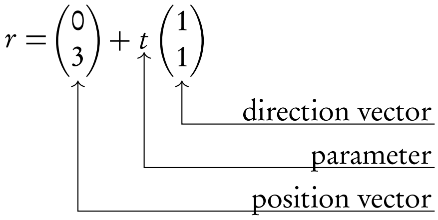
We can further divide vectors into two types:

position vectors vectors from the origin to a point,

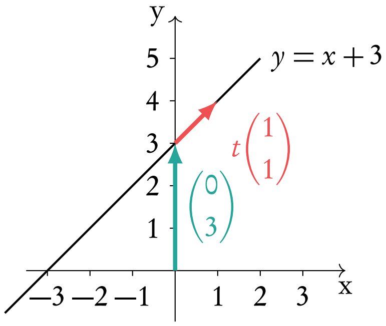
e.g. $P = (-1,3) \Rightarrow \vec{P} = \begin{pmatrix}-1\\3\end{pmatrix}$.

direction vectors vectors that define a direction.

Using both we can define lines in terms of vectors.
Example of a line:



$$r = \begin{pmatrix}0\\3\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$$



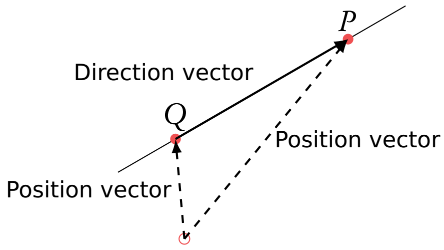
Note the position vector can go to any where on the line. So in this example we could also use $(-3,0)$ or $(1,4)$. Equally the direction vector can be scaled. So we could use $(2,2)$, $(30,30)$, …

Because of this parallel lines will have direction vectors with the same ratio but not necessarily in exact numbers.

Parallel lines: direction vector of L_1 = direction vector of $L_2 \times$ constant

Questions often deal with points and or multiple lines. It is worth making a sketch to help understand the question.

Finding a line passing through two points.



Find the equation of the line passing through points $P = (1,3,2)$ and $Q = (0,-1,4)$. Does point $R = (-2,9,1)$ lie on the line?

Note this can go either way from Q to P of P to Q.

1. Write points as position vectors

$$\vec{P} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \vec{Q} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

2. Direction vector
= vector between points

$$\begin{pmatrix} 0-1 \\ -1-3 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

3. Choose $\vec{P}$ or $\vec{Q}$ as position vector

$$r = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

4. Equate $\vec{R}$ and the line r.
If there is no contradiction,
R lies on r

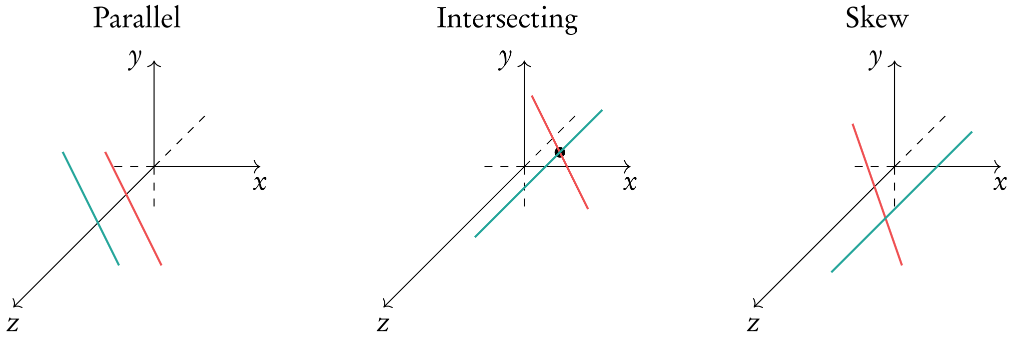
$$\begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

$$\Rightarrow -2 = 1 - t \Rightarrow t = 3$$

$$\Rightarrow 9 = 3 - 4t \Rightarrow 9 \neq 3 - 12$$

$\Rightarrow R$ does not lie on the line.

If one considers two lines in a three-dimensional graph, then there are three ways in which they can interact:


Parallel
Intersecting
Skew

If direction vectors defining a line aren't multiples of one another, then the lines can either be intersecting or skew. One can find out if the lines intersect by equating the vector equations and attempting to solve the set of equations (remember: one needs as many equations as variables to solve).

If one can't find a point of intersection, then the lines are skew.

Finding the intersection of two lines.

Find the intersection for $r_1 = \begin{pmatrix}2\\1\\0\end{pmatrix} + s\begin{pmatrix}-3\\1\\4\end{pmatrix}$ and $r_2 = \begin{pmatrix}-1\\3\\7\end{pmatrix} + t\begin{pmatrix}3\\0\\-1\end{pmatrix}$

1. Equate
write simultaneous equations

$$\begin{cases} 2-3s = -1+3t \\ 1+s = 3 \end{cases}$$

2. Solve

$$s = 2,\ t = -1$$

3. Substitute back into r_1 or r_2

$$\begin{pmatrix}2-3(2)\\1+2\\4(2)\end{pmatrix} = \begin{pmatrix}-4\\3\\8\end{pmatrix}$$

3.3 Dot (scalar) product

DB 4.2

The dot product of two vectors $\vec{c}\cdot\vec{d}$ can be used to find the angle between them.

$$\vec{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \qquad \vec{d} = \begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$$

$$\vec{c}\cdot\vec{d} = |\vec{c}||\vec{d}|\cos\vartheta$$

$$\vec{c}\cdot\vec{d} = c_1d_1 + c_2d_2 + c_3d_3$$

Finding the angle between two lines.
(Often are these two vectors perpendicular)

Find the angle between $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}8\\1\\3\end{pmatrix}$.

1. Find $\vec{c}\cdot\vec{d}$ in terms of components

$$\vec{c}\cdot\vec{d} = 2\times 8 + 3\times 1 + (-1)\times 3 = 16$$

2. Find $\vec{c}\cdot\vec{d}$ in terms of magnitudes

$$\vec{c}\cdot\vec{d} = \sqrt{2^2+3^2+(-1)^2}\times\sqrt{8^2+1^2+3^2}\times \cos\vartheta = \sqrt{14}\sqrt{74}\cos\vartheta$$

3. Equate and solve for ϑ

$$16 = \sqrt{14}\sqrt{74}\cos\vartheta \Rightarrow \cos\vartheta = \frac{16}{\sqrt{14}\sqrt{74}}$$

$$\Rightarrow \vartheta = 60.2^\circ$$

When $\vartheta = 90°$ the vectors are perpendicular. As $\cos(90°) = 0 \Rightarrow \vec{c}\cdot\vec{d} = 0$ Learn to add the following statement to questions asking "*are they perpendicular?*".
$\vec{c}\cdot\vec{d} = 0$ therefore $\cos x = 0$, therefore $x = 90°$. Lines are perpendicular. Of course, when lines are not perpendicular replace all $=$ with $\neq$.

3.4 Cross (vector) product

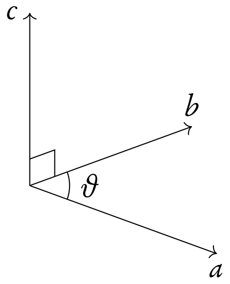
The cross product of two vectors produces a third vector which is perpendicular to both of the two vectors. As the result is a vector, it is also called the *vector product*.

There are two methods to find the cross product:

1. $a \times b = |a||b| \sin \vartheta \mathrm{n}$
   where ϑ is the angle between a and b and n is a unit vector in the direction of c.

2. $x = a \times b$, where

$$c_1 = a_2 b_3 - a_3 b_2$$
$$c_2 = a_3 b_1 - a_1 b_3$$
$$c_3 = a_1 b_2 - a_2 b_1$$



Example.

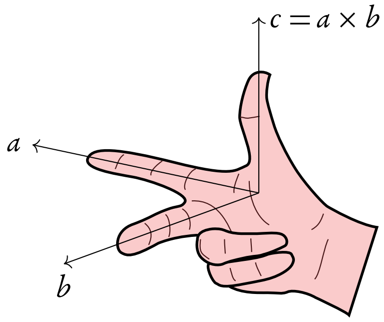
Find the cross product of $a \times b$.
$a = (2, 3, 4)$, $b = (5, 6, 7)$.

$$c_1 = 3 \times 7 - 4 \times 6 = -3$$
$$c_2 = 4 \times 5 - 2 \times 7 = 6$$
$$c_3 = 2 \times 6 - 3 \times 5 = -3$$

$\Rightarrow a \times b = (-3, 6, -3)$

Remember the cross product is not commutative, so $a \times b \neq b \times a$.

You can check the direction of c with the right hand rule:



3.4.1 Definition and properties

There are several important properties of cross product that can be shown for vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Some of them are useful to memorise, while the rest can be found with right hand rule or by using the definition of a vector product.

$$\begin{aligned} \vec{a}\times\vec{b} &= -\left(\vec{b}\times\vec{a}\right) \\ \vec{a}\times\left(\vec{b}\pm\vec{c}\right) &= \vec{a}\times\vec{b}\pm\vec{a}\times\vec{c} \\ \vec{a}\times 0 &= 0 \end{aligned}$$

$$\begin{aligned} \vec{a}\times\vec{a} &= 0 \\ \vec{a}\cdot\left(\vec{a}\times\vec{b}\right) &= 0 \\ \vec{b}\cdot\left(\vec{a}\times\vec{b}\right) &= 0 \end{aligned}$$

$$\begin{aligned} \vec{i}\times\vec{j} &= \vec{k} \\ \vec{j}\times\vec{k} &= \vec{i} \\ \vec{k}\times\vec{i} &= \vec{j} \end{aligned}$$

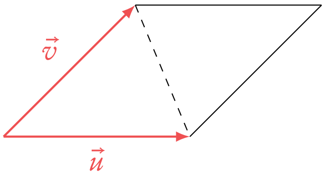
3.4.2 Geometric interpretation of vector product

The length of the cross product can be found by either of the two methods:

1. $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\vartheta)$ where ϑ is the angle between vectors $\vec{u}$ and $\vec{v}$.
2. By calculating the vector with use of cross product formula and then finding the length of that vector.

However, there are two main interpretations of length of the vector product:

1. The length of the vector, that you get from cross product of two vectors.
2. Area of a parallelogram made up from the two vectors.

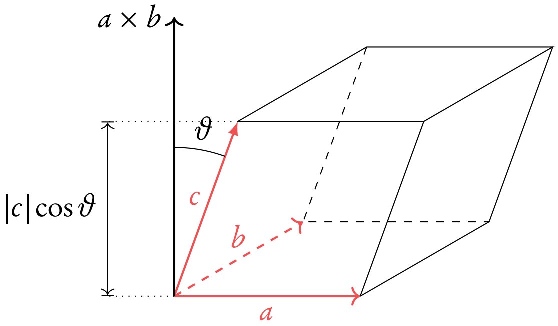


It also means, that half of the length of the cross product will be the area of triangle made up from the two original vectors.

However, the vector product can also be used when finding the volume of parallelepiped, that is made up from three vectors. Usually, to find its volume, we need base × height. Base can be found with use of the vector product. Thus we get the following formula:

The volume of parallelepiped spanned with use of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

$$|(\vec{a} \times \vec{b}) \cdot \vec{c}|$$



Example.

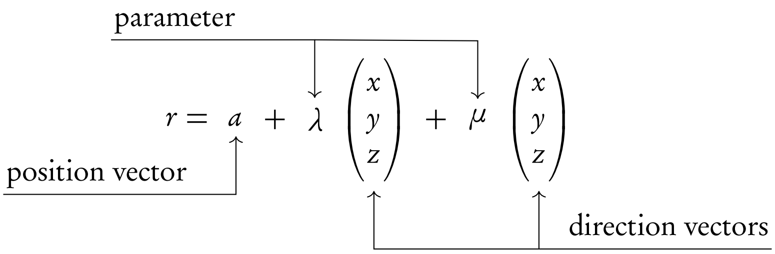
Find the area of triangle with sides $\vec{a} = (1, 3, 5)$ and $\vec{b} = (-1, 2, 3)$.

$$\begin{aligned} c_1 &= 3 \times 3 - 5 \times 2 = -1 \\ c_2 &= 5 \times -1 - 1 \times 3 = -8 \\ c_3 &= 1 \times 2 - 3 \times -1 = 5 \\ \vec{a} \times \vec{b} &= (-1, -8, 5) \\ |\vec{a} \times \vec{b}| &= |(-1, -8, 5)| = \sqrt{1 + 64 + 25} = 3\sqrt{10} \end{aligned}$$

Thus the area of triangle is: $1.5\sqrt{10}$

3.5 Equation of a plane

Planes are 2 dimensional surface in 3 dimensional space. They can be defined by a position vector and 2 direction vectors (which are not parallel)



$$r = a + \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mu \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or in a cartesian form

$$ax + by + cz = d$$

where d is a constant.

Find a cartesian equation of a plane from 3 points.

Find a Cartesian equation of the plane P containing $A(2,0,-3)$, $B(1,-1,6)$ and $C(5,5,0)$.

1. Find two lines

$$AB = B - A = -i - j + ak$$
$$AC = C - A = 3i + 5j + 3k$$

2. Take the cross product of these two lines

$$AB \times AC = -48i + 30j - 2k$$

3. Substitute a point back in to the cross product (here A)

$$-48(x-2) + 30(y) - 2(z+3) = 0$$
$$-48x + 30y - 2z = -90$$
$$24x - 15y + z = 45$$

3.5.1 Line and plane

Lines can intersect with a plane in 3 ways:

1. Parallel to the plane	0 solutions
2. Intersect the plane	1 solution
3. Lie on the plane	infinite solutions

Does a line intersect a plane?

The line L_1 passes through the points $(1, 0, 1)$ and $(4, -2, 2)$. Does it intersect the plane $x + y + 2 = 6$.

1.	Find parameter representation of the line	$x = 1 + 3\lambda$ $y = -2\lambda$ $z = 1 + \lambda$
2.	Put into the equation for the plane	$(1 + 3\lambda) + (-2\lambda) + (1 + \lambda) = 6$
3.	Solve for λ	$2 + 2\lambda = 6$ $2\lambda = 4$ $\lambda = 2$
4.	Find point of intersection	$x = 1 + 3(2) = 7$ $y = -2(2) = -4$ $z = 1 + (2) = 3$ $\begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} =$ point of intersection

3.5.2 Plane and plane

Intersection of two planes

When two planes intersect, they will intersect along a line.

Finding line of intersection of two planes

Find the intersection of $x+y+z+1=0$ and $x+2y+3z+4=0$

1.	Check the equations for planes are in a Cartesian form; move the constant to the other side	$\begin{cases} x+y+z=-1 & (1) \\ x+2y+3z=-4 & (2) \end{cases}$
2.	Solve the system of equations to remove a variable	(1) − (2) $\begin{array}{r} \begin{cases} x+\ y+\ z=-1 \\ -x-2y-3z=-4 \end{cases} \\ \hline -y-2z=\ \ 3 \end{array}$ $y=-3-2z$
3.	Let $z=t$	$\Rightarrow \quad y=-3-2t$. Rearrange: $x=-1-y-z$ $x=-1-(-3-2t)-t$ $x=t+2$
4.	Find the result	Intersection occurs at line $(x,y,z)=(t+2,-2t-3,t)$ or $r=\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}+\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}t$

Intersection of three planes

Unless two or more planes are parallel, three planes will intersect at a point. If two are parallel there will be two lines of intersect. If all three are parallel, there will be no solutions.

We have three variables and three equations and so we can solve the system.

Finding point of intersect of three planes

Find the intersect of the three planes

$$\begin{aligned} x-3y+3z&=-4 && \text{(a)}\\ 2x+3y-z&=15 && \text{(b)}\\ 4x-3y-z&=19 && \text{(c)} \end{aligned}$$

1.	Eliminate one variable in two pair of lines (here z)	(b) − (c) $\Rightarrow$ $-2x+6y=-4$ (d) (a) + 3(b) $\Rightarrow$ $7x+6y=41$ (e)
2.	Eliminate another variable from these new lines (here y)	(e) − (d) $\Rightarrow$ $9x=45$ $x=5$
3.	Place the value into the equations to find values for x, y, and z	$x=5$ (d) $-2(5)+6y=-4 \Rightarrow y=1$ (a) $(5)-3(1)+3z=-4 \Rightarrow z=-2$ Point of intersection $(5,1,-2)$

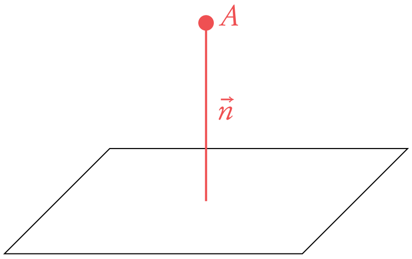
3.5.3 Normal vector

By taking the cross product of the two direction vectors that define a plane, we can find the normal vector. This vector is perpendicular to the plane. In turn the normal vector can be used to show a line is parallel to the plane by using the dot product. If parallel $\mathrm{n} \cdot d = 0$, where n is the normal vector and d is the direction vector of a line.

Use of normal vector

Normal vector can also be used to determine closest distance from a point or a parallel line to the plane. Since the line is parallel, any point will be equal distance away from the plane. So take any point on the line and proceed as if you were trying to find distance from a point to a plane.

To find the distance to the plane from a point, we need a line, that goes through the point and is perpendicular to the plane. Then, we need to find a point on the line, that lies on the plane. Finally, you can simply find the distance between those two points, which will give you the final answer.



Example.

Find distance from a point $A = (3, 2, 5)$ to the plane with equation $2x + 3y - 4z = 6$.

Normal vector equation: $\vec{n} = (2, 3, -4)$. Thus our line is:

$$r = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

Thus, the coordinate of a point B that is on the plane and the line is:

$$x = 3 + 2t \qquad y = 2 + 3t \qquad z = 5 - 4t$$

However, it should also satisfy the original plane equation $2x + 3y - 4z = 6$:

$$2(3 + 2t) + 3(2 + 3t) - 4(5 - 4t) = 6$$

$$t = \frac{14}{29}$$

Thus our point $B = \left(\frac{115}{29}, \frac{100}{29}, \frac{89}{29}\right)$, meaning that the distance from the point to the plane is ≈ 2.60.

3.6 Angles between: line and plane; two planes

Finding an angle between two vectors is fairly easy: simply use the dot product. However, it raises a question on how to find an angle between two planes or a line and a plane.

Finding an angle between two planes is easier. To do that, simply write planes in Cartesian form and find an angle between normal vectors of those planes. Since both normal vectors are perpendicular to their corresponding planes, the overall effect of the vectors being 90° turned cancels out.

To find an angle between a line and a plane, use the dot product to find an angle between directional vector of a line and normal vector of the plane. Since the normal vector is perpendicular to the plane, subtract the gotten result from 90°.

Example.

Find an angle between plane $2x+3y-z=4$ and line $r=\begin{pmatrix}2\\3\\4\end{pmatrix}+t\begin{pmatrix}3\\1\\-2\end{pmatrix}$.

First we find an angle between normal vector and the directional vector of the line:

$$\cos(\vartheta)=\frac{2\times 3+3\times 1+2\times 1}{\sqrt{4+9+1}\sqrt{9+1+4}}$$

$$\vartheta=38.21°$$

Since it is a line and a plane, subtract the gotten result from 90°:

$$\alpha=90°-38.21°$$

$$\alpha=51.8°$$

4

TRIGONOMETRY AND CIRCULAR FUNCTIONS

Table of contents & cheatsheet

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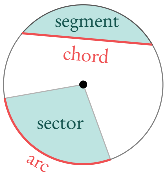
$\text{radians} = \dfrac{\pi}{180°} \times \text{degrees}$ $\text{degrees} = \dfrac{180°}{\pi} \times \text{radians}$

Before each question make sure calculator is in correct setting: degrees or radians?

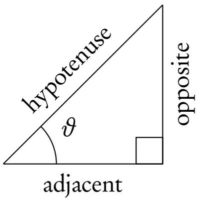


Area of a sector $= \dfrac{1}{2} r^2 \cdot \vartheta$

Arc length $= r \cdot \vartheta$

ϑ in radians, $r =$ radius.

Right-angle triangle (triangle with 90° angle)

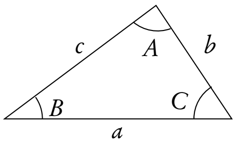


$\sin\vartheta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ **SOH**

$\cos\vartheta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ **CAH**

$\tan\vartheta = \dfrac{\text{opposite}}{\text{adjacent}}$ **TOA**

Non-right angle triangles



Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Use this rule when you know: 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

Cosine rule: $c^2 = a^2 + b^2 - 2\,ab\cos C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

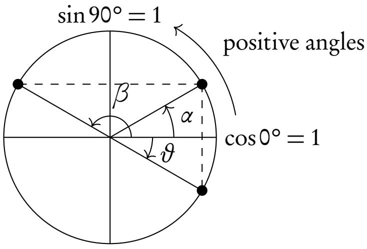
Area of a triangle: $\text{Area} = \dfrac{1}{2} ab \sin C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

Three-figure bearings

Direction given as an angle of a full circle. North is 000° and the angle is expressed in the clockwise direction from North. So East is 090°, South is 180° and West 270°.

4.3. Circular functions 71



deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
rad	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
sin ϑ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos ϑ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan ϑ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Trigonometric function $y = a\sin(bx + c) + d$

Amplitude: a

Period: $\dfrac{360°}{b}$ or $\dfrac{2\pi}{b}$

Horizontal shift: c

Vertical shift: d

Trigonometric identities

$$\tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta}$$

$$\sin^2\vartheta + \cos^2\vartheta = 1$$

$$2\sin\vartheta\cos\vartheta = \sin 2\vartheta$$

$$\cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta$$

4.1 Properties of 3D shapes

4.1.1 Points in 3D space

When you are learning about the points on a 2-dimensional plane, you also learn how to find distances between those two points. One of the easiest ways to derive that formula is by constructing a triangle and using Pythagoras. In the same way it is possible to derive a very similar expression for distance between two points in a 3D space:

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

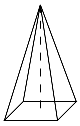
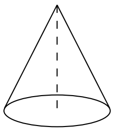
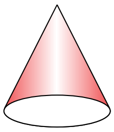
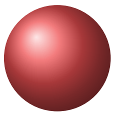
You have also learnt how to find the midpoint between the two points (x_1,y_1) and (x_2,y_2): add those individual coordinates together and divide the sum by two. One can find the midpoint between two points in 3D space in almost exact same way:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

4.1.2 Pyramids, cones and spheres

In the exam you might be asked to find volume or surface of different 3D objects. These objects might be either familiar to you 3D shapes or made up from those shapes. In the first case, the formulas should be given in the data booklet. In the latter case, you would need to split the object into familiar shapes and sum the required components together.

DB 3.1

Volume of a right-pyramid	$V=\frac{1}{3}Ah$	
Volume of a right cone	$V=\frac{1}{3}\pi r^2h$	
Area of the curved surface of a cone	$A=\pi rl$	
Volume of a sphere	$V=\frac{4}{3}\pi r^3$	
Surface area of a sphere	$A=4\pi r^2$	

4.2 Basic trigonometry

This section offers an overview of some basic trigonometry rules and values that will recur often. It is worthwhile to know these by heart; but it is much better to understand how to obtain these values. Like converting between Celsius and Fahrenheit; you can remember some values that correspond to each other but if you understand how to obtain them, you will be able to convert any temperature.

4.2.1 Converting between radians and degrees

$$\text{radians} = \frac{\pi}{180°} \times \text{degrees}$$

$$\text{degrees} = \frac{180°}{\pi} \times \text{radians}$$

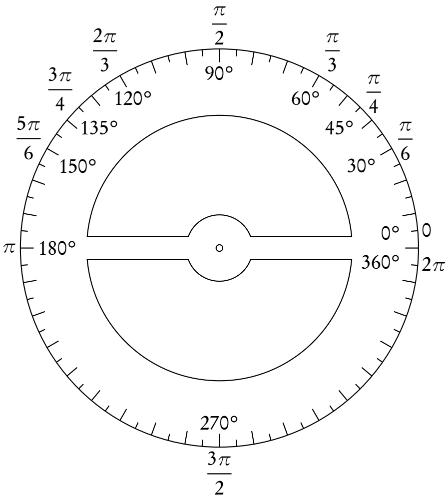


Table 4.1: Common radians/degrees conversions

Degrees	0°	30°	45°	60°	90°	120°	135°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

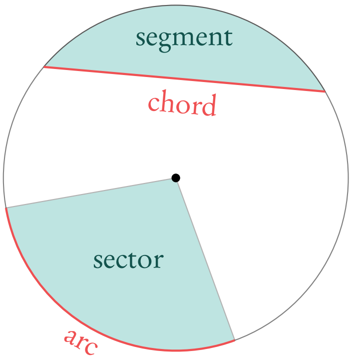
4.2.2 Circle formulas

DB 3.4

Area of a sector $= \frac{1}{2} r^2 \cdot \vartheta$

Arc length $= r \cdot \vartheta$

ϑ in radians, $r =$ radius.



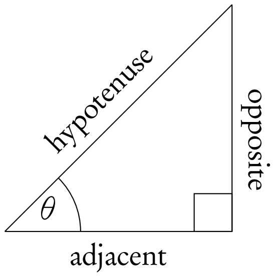
4.2.3 Right-angle triangles

$$a^2 = b^2 + c^2 \qquad \text{Pythagoras}$$

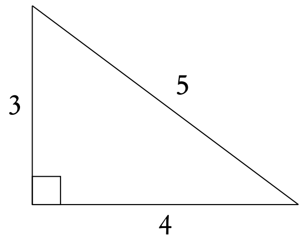
$$\sin\vartheta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{SOH}$$

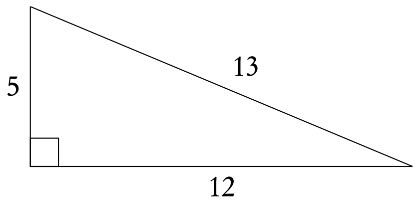
$$\cos\vartheta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{CAH}$$

$$\tan\vartheta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{TOA}$$

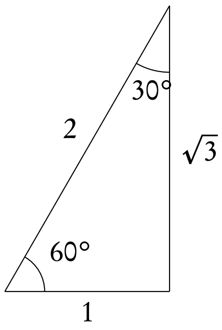


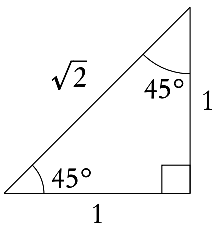
The following two right angle triangles with whole numbers for all the sides come up often in past exam questions.





The two triangles below can help you in finding the sin, cos and tan of the angles that you should memorize, shown in table 4.2 on page 72. Use SOH, CAH, TOA to find the values.

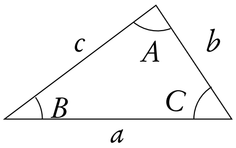




4.2.4 Non-right angle triangles

Read the question: does it specify if you are looking for an acute (less than 90°) or obtuse (more than 90°) angle? If not there may be 2 solutions. Exam hint: Use sketches when working with worded questions!

DB 3.2

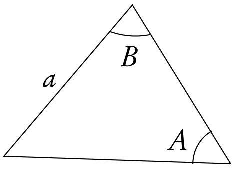


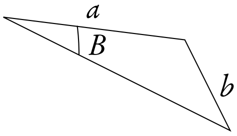
To find any missing angles or side lengths in non-right angle triangles, use the *cosine* and *sine* rule. Remember that the angles in the triangle add up to 180°!

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Use this rule when you know:

2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides)

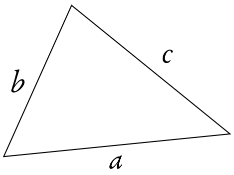


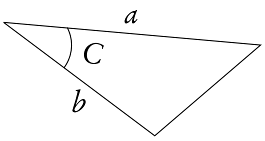


Cosine rule: $c^2 = a^2 + b^2 - 2\,ab\cos C$

Use this rule when you know:

3 sides or 2 sides and the angle between them

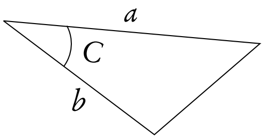




Area of a triangle: $\text{Area} = \dfrac{1}{2}ab\sin C$

Use this rule when you know:

2 sides and the angle between them



Example.

$\triangle ABC$: $A = 40°$, $B = 73°$, $a = 27\,\text{cm}$.

Find $\angle C$

$$\angle C = 180° - 40° - 73° = 67°$$

Find b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{27}{\sin 40°} = \frac{b}{\sin 73°}$$
$$b = \frac{27}{\sin 40°} \cdot \sin 73° = 40.169 \approx 40.2\,\text{cm}$$

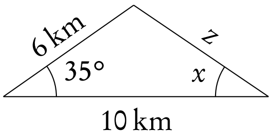
Find c

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{27}{\sin 40°} \times \sin 67° = 38.7\,\text{cm}$$

Find the area

$$\text{Area} = \frac{1}{2} \cdot 27 \cdot 40 \cdot 2 \cdot \sin 67°$$
$$= 499.59 \approx 500\,\text{cm}^2$$

Example.



Find z

$$z^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35°$$
$$z^2 = 37.70$$
$$z = 6.14\,\text{km}$$

Find $\angle x$

$$\frac{6}{\sin x} = \frac{6.14}{\sin 35°}$$
$$\sin x = 0.56$$
$$x = \sin^{-1}(0.56) = 55.91°$$

4.2.5 Ambiguous case

Ambiguous case, also known as an angle-side-side case, is when the triangle is not unique from the given information. It happens when you are given two sides and an angle not between those sides in a triangle.

You have to use a sine rule to solve a problem in this case. However, one needs to remember that $\sin x = \sin(180° - x)$, meaning that your answer for an angle is not just x, but also $180° - x$.

In other words, we might get two different possible angles as an answer and thus two different possible triangles that satisfy the information given.

However, that is not always the case, if the sum of the two known angles becomes bigger than 180°. So if you are required to calculate the third angle or total area of a triangle, you might have to do the calculations for two different triangles using both of your angles.

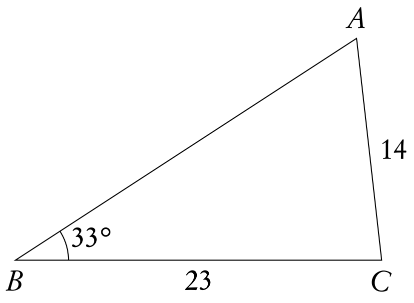
Example.

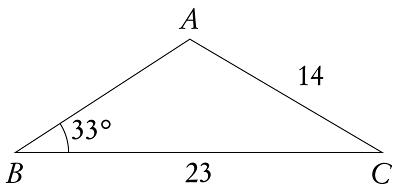
$\triangle ABC : B = 33°$, $a = 23\,\text{cm}$, $b = 14\,\text{cm}$.

Find $\angle A$.

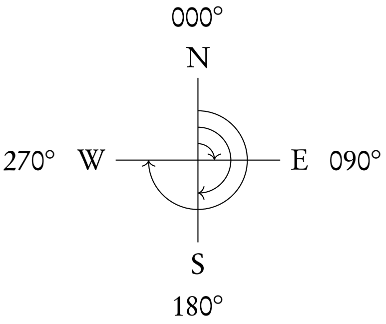
$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B}\\
\frac{14}{\sin 33°} &= \frac{23}{\sin A}\\
\angle A_1 &= 63.5°\\
\angle A_2 &= 180° - 63.5° = 117°\\
\angle A_2 + 33° &< 180° \text{ thus also a possible angle}
\end{aligned}$$

Draw the two possible triangles.





4.2.6 Three-figure bearings

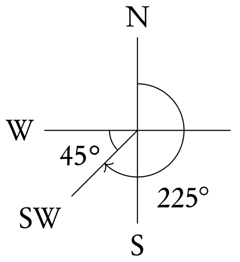


Three-figure bearings can be used to indicate compass directions on maps. They will be given as an angle of a full circle, so between 0° and 360°. North is always marked as 0°. Any direction from there can be expressed as the angle in the clockwise direction from North.

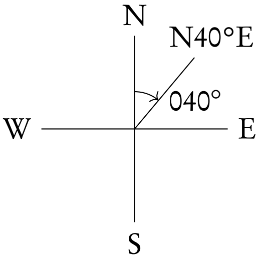
In questions on three-figure bearings, you are often confronted with quite a lot of text, so it is a good idea to first make a drawing. You may also need to create a right angle triangle and use your basic trigonometry.

Example.

SW: 45° between South and West = 225°



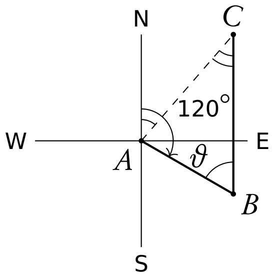
N40°E: 40° East of North = 040°



Solving questions with three-figure bearings

A ship left port A and sailed 20 km in the direction 120°. It then sailed north for 30 km to reach point C. How far from the port is the ship?

1. Sketch



2. Find an internal angle of the triangle

$\vartheta = 180^\circ - 120^\circ = 60^\circ = C$

Similar angles between two parallel lines

3. Use cosine or sin rule

(here cosine)

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\vartheta$$

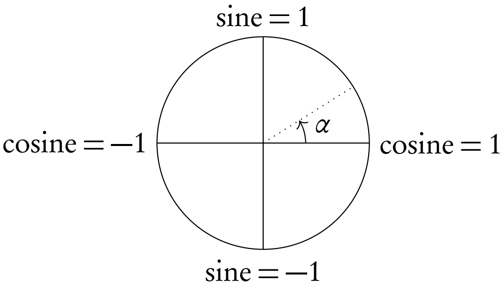
$$AC^2 = 20^2 + 30^2 - 2 \cdot 20 \cdot 30 \cdot \cos 60^\circ$$

$$AC^2 = 400 + 900 - 2 \cdot 20 \cdot 30 \cdot \frac{1}{2}$$

$$AC = \sqrt{400 + 900 - 600} = \sqrt{700}$$

4.3 Circular functions

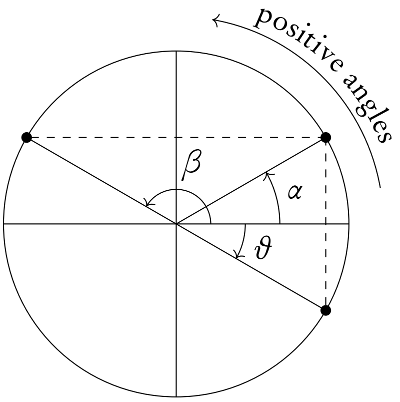
4.3.1 Unit circle



The unit circle is a circle with a radius of 1 drawn from the origin of a set of axes. The y-axis corresponds to *sine* and the x-axis to *cosine*; so at the coordinate $(0, 1)$ it can be said that cosine $= 0$ and sine $= 1$, just like in the $\sin x$ and $\cos x$ graphs when plotted.

The unit circle is a a tool that you can use when solving problems involving circular functions. You can use it to find all the solutions to a trigonometric equation within a certain domain.

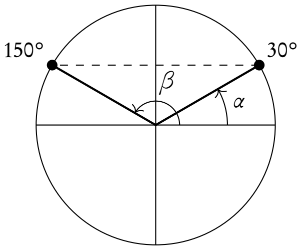
As you can see from their graphs, functions with $\sin x$, $\cos x$ or $\tan x$ repeat themselves every given period; this is why they are also called *circular functions*. As a result, for each y-value there is an infinite amount of x-values that could give you the same output. This is why questions will give you a set domain that limits the x-values you should consider in your calculations or represent on your sketch (e.g. $0° \leq x \leq 360°$).

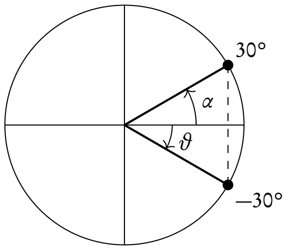


Relations between sin, cos and tan:

- α and β have the same sine
- α and ϑ have the same cosine
- β and ϑ have the same tangent

Example.


$\sin 30° = \sin 150°$


$\cos 30° = \cos 330°$

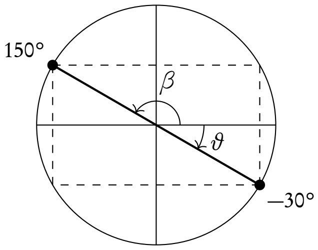
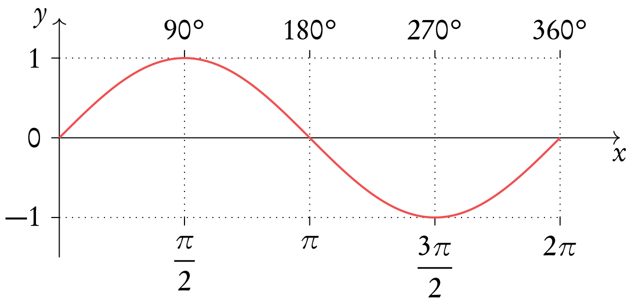

$\tan 150° = \tan 330°$

Table 4.2: Angles to memorize

deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
rad	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
sin ϑ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos ϑ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan ϑ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

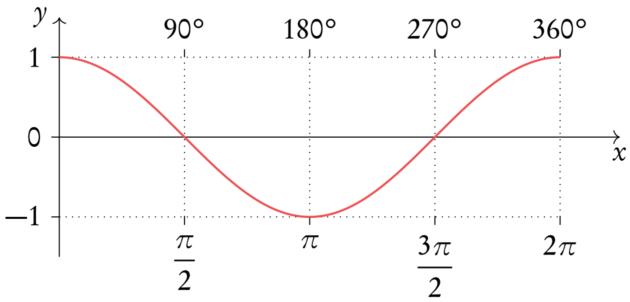
4.3.2 Graphs of trigonometric functions

sin x



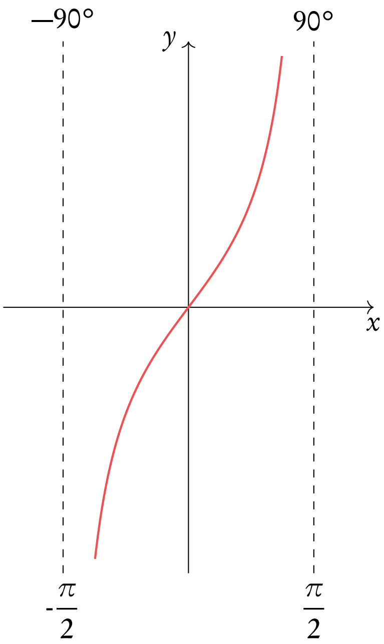
Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

cos x



Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

tan x



Domain: $x \in \mathbb{R}$, $x \neq \frac{\pi}{2} + k\pi$, with $k \in \mathbb{Z}$
Amplitude: $-\infty < y < \infty$
Period: $n\pi$, $n \cdot 180°$, with $n \in \mathbb{Z}$

4.3.3 Transformations

Besides the transformations in the functions chapter, trigonometric functions have some transformations with their own particular names. For a trigonometric function, the vertical stretch on a graph is determined by its amplitude, the horizontal stretch by its period and an upward/downward shift by its axis of oscillation.

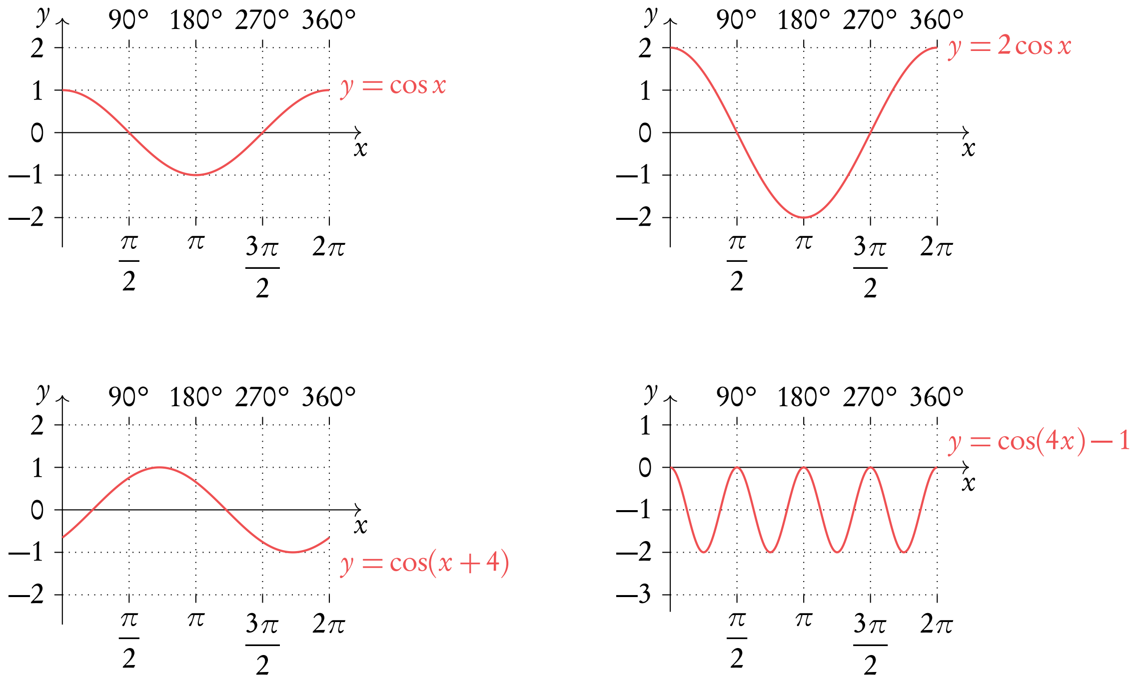
A trigonometric function, given by $y = a\sin(bx+c)+d$, has:

- amplitude a
- period of $\frac{360°}{b}$ or $\frac{2\pi}{b}$
- horizontal shift of $+c$ *to the left*, in degrees or radians
- vertical shift of $+d$ *upwards*, oscillates around d.

A negative a will flip your graph around the x-axis. Negative values of c and d will lead to shifts to the right and downwards the respective number of units

Example.

Transformations of $y = \cos x$.



4.3.4 Identities and equations

DB 3.5 & 3.6

In order to solve trigonometric equations, you will sometimes need to use identities. Identities allow you to rewrite your equation in a way that will make it easier to solve algebraically.

Trigonometric identity

$$\tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta}$$
$$\sin^2\vartheta + \cos^2\vartheta = 1$$
$$2\sin\vartheta\cos\vartheta = \sin 2\vartheta$$
$$\cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta$$

Solving equations with trigonometric identities

Solve $2\cos^2 x + \sin x = 1$, $0^\circ \le x \le 360^\circ$.

1.	Identify which identity from the databook to use. Note you are always aiming to get an equation with just, sin, cos or tan.	Here we could use either $\sin^2\vartheta + \cos^2\vartheta = 1$ or $\cos^2\vartheta - \sin^2\vartheta = \cos 2\vartheta$. We will use the first so that we get an equation with just sin.
2.	Rearrange identity and substitute into equation.	$\cos^2\vartheta = 1 - \sin^2\vartheta$ $2(1-\sin^2 x) + \sin x = 1$ $2 - 2\sin^2 x + \sin x = 1$ $-2\sin^2 x + \sin x + 1 = 0$
3.	Solve for x. Giving answers within the stated range. Recognise that here the equation looks like a quadratic equation.	Substitute u for $\sin x$: $-2u^2 + u + 1 = 0$ $(-2u-1)(u-1) = 0$ $u = \sin x \Rightarrow 1$ $\quad x \Rightarrow 90^\circ$ $u = \sin x \Rightarrow -0.5$ $\quad x \Rightarrow 210^\circ$ or 330°

Double angle and half angle formulae

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \\
\cos(2a) &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\
\sin(2a) &= 2\sin a \cos a \\
\tan(2a) &= \frac{2\tan a}{1 - \tan^2 a}
\end{aligned}
$$

From the double angle we can obtain half angles.

$$
\begin{aligned}
\cos a = \cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right) = 2\cos^2\left(\frac{a}{2}\right) - 1 = 1 - 2\sin^2\left(\frac{a}{2}\right) \\
\sin a = 2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right) \\
\tan a = \frac{2\tan\left(\frac{a}{2}\right)}{1 - \tan^2\left(\frac{a}{2}\right)}
\end{aligned}
$$

4.3.5 Inverse and reciprocal trigonometric functions

DB

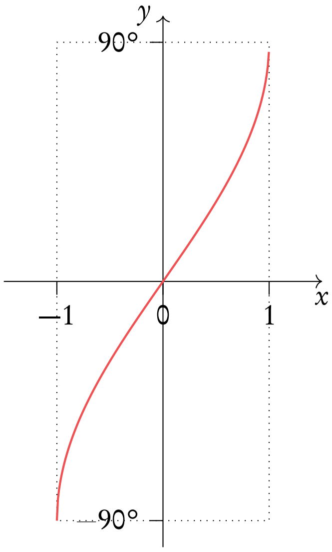
Inverse trigonometric functions

The inverse of a trigonometric function is useful for finding an angle. You should already be familiar with carrying this operation out on a calculator.

$$\sin\vartheta = \frac{\pi}{2} \quad \Rightarrow \quad \vartheta = \arcsin\frac{\pi}{2}$$

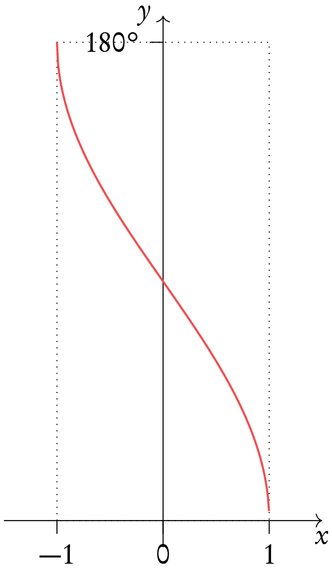
Just like the inverse functions, trigonometric inverse functions have the property that the range of the original function is its domain and vice versa.

$\sin^{-1} x = \arcsin x$



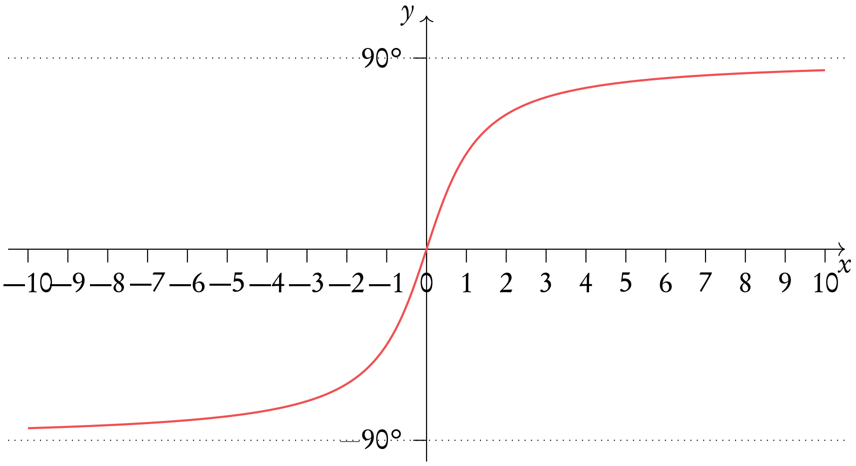
Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\cos^{-1} x = \arccos x$



Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

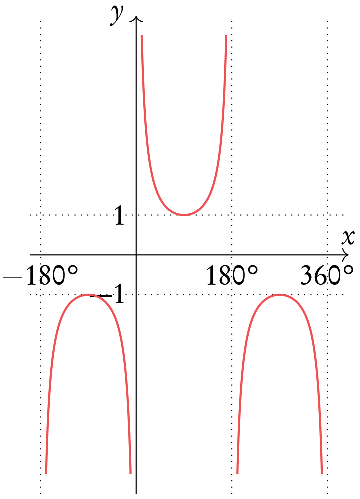
$\tan^{-1} x = \arctan x$



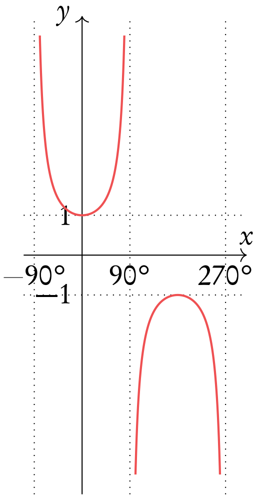
Domain: $x \in \mathbb{R}$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Reciprocal trigonometric functions

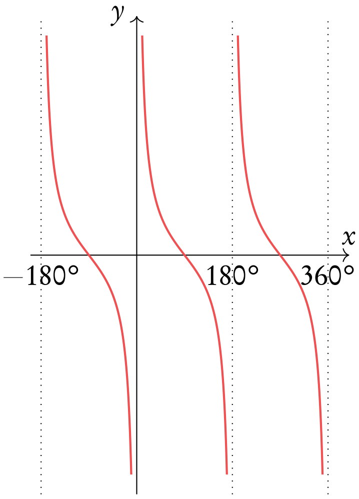
$$\frac{1}{\sin\vartheta} = \csc\vartheta$$



$$\frac{1}{\cos\vartheta} = \sec\vartheta$$



$$\frac{1}{\tan\vartheta} = \cot\vartheta$$



These functions are the reciprocal functions, their vertical asymptotes correspond to the x-axis intercepts of the original function. The functions $\csc\vartheta$ and $\sec\vartheta$ are periodic with a period of 360°, $\cot\vartheta$ has a period of 180°.

5

DIFFERENTIATION

Table of contents & cheatsheet

Definitions

Differentiation is a way to find the gradient of a function at any point, written as $f'(x)$, y' and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Tangent line to a point on a curve is a linear line with the same gradient as that point on the curve.

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Product $y = uv$, then: $y' = uv' + u'v$

Quotient $y = \frac{u}{v}$, then: $y' = \frac{vu' - uv'}{v^2}$

Chain $y = g(u)$ where $u = f(x)$, then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

5.4. Tangent and normal 87

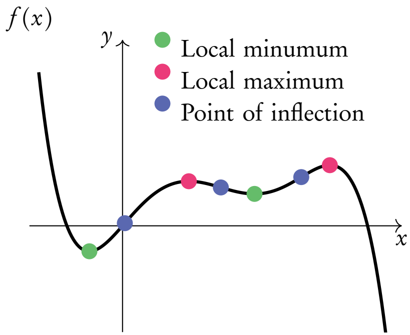
Tangent line with the same gradient as a point on a curve.

Normal perpendicular to the tangent $m = \frac{-1}{\text{slope of tangent}}$

Both are linear lines with general formula: $y = mx + c$.

1. Use derivative to find gradient of the tangent. For normal then do $-\frac{1}{\text{slope of tangent}}$.
2. Input the x-value of the point into $f(x)$ to find y.
3. Input y, m and the x-value into $y = mx + c$ to find c.

5.5. Turning points 88



	$f'(x)$	$f''(x)$
Local minimum	0	+
Local maximum	0	−
Points of inflection		0

5.6. Sketching graphs 91

Gather information before sketching:

Intercepts
x-intercept: $f(x) = 0$
y-intercept: $f(0)$

Turning points
minima: $f'(x) = 0$ and $f''(x) < 0$
maxima: $f'(x) = 0$ and $f''(x) > 0$
point of inflection: $f''(x) = 0$

Asymptotes
vertical: x-value when the function divides by 0
horizontal: y-value when $x \to \infty$

Plug the found x-values into $f(x)$ to determine the y-values.

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Kinematics

Derivative represents the rate of change, integration the reverse.

$$\begin{tikzcd} s \arrow[r, bend left, "\frac{\mathrm{d}s}{\mathrm{d}t} = v"] & v \arrow[r, bend left, "\frac{\mathrm{d}v}{\mathrm{d}t} = a"] & a \\ \text{displacement} & \text{velocity} & \text{acceleration} \end{tikzcd}$$

5.1 Limits

Limits are a great tool to find which value a function approaches as its input approaches some value. In other words, using limits we can find which value function approaches, even if it is not defined at that point. Let's take a look at an example:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}$$

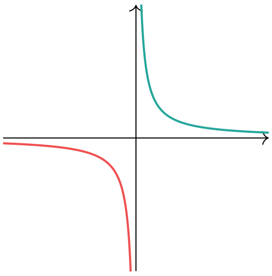
If we try to obtain the value of the function at $x=1$, we quickly find out that we try to calculate zero divided by zero. There is no defined answer to that question, that is why it is called "indeterminate". However, we can find what value the function approaches as x approaches 1. First we need to simplify the expression and then we can plug in the value for x to evaluate the limit:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}=\lim_{x\to 1}x+1=1+1=2$$

If we draw a function we can see that it approaches point $(1,2)$, even if the function is actually undefined at $x=1$. It is important to note, that the limit exists only if the function approaches it from both positive and negative sides. As an example we need to evaluate the following function as x approaches 0:

$$\lim_{x\to 0}\frac{1}{x}$$

We cannot plug in $x=0$ since we cannot divide by zero directly. Let's try to first see what values the function approaches from the positive (right) and negative (left) sides:



As it can be seen from positive side the function diverges towards positive infinity. We can write that down in the following way (note the small plus sign near the zero, indicating that we are calculating from the positive side):

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

However, from the negative side the function diverges towards negative infinity:

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty$$

The negative and positive limits do not equal to each other, so the limit does not exist:

$$\lim_{x\to 0}\frac{1}{x}=\text{DNE}$$

5.1.1 L'Hôpital's rule

Sometimes we need to find the limit of a function but we are unable to plug in the value for x directly because we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If it is also impossible to simplify the expression, then L'Hôpital's rule is required to be used.

L'Hôpital's rule if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Example.

Evaluate the following limit: $\lim_{x \to \infty} \frac{e^x}{x^2}$.

We quickly figure out that the expression is of the form $\frac{\infty}{\infty}$, so we use L'Hôpital's rule and differentiate top and bottom expressions:

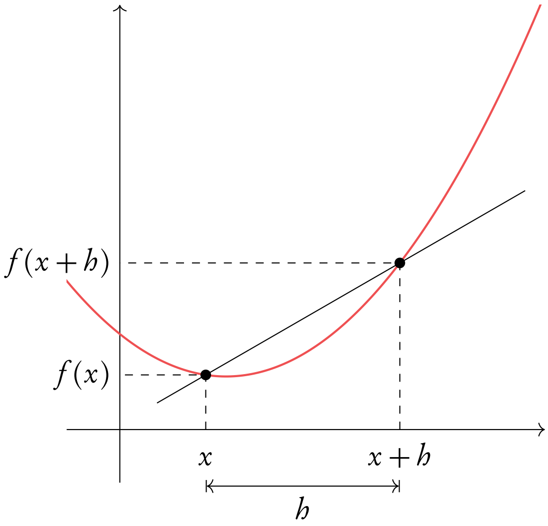
$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$$

After doing the L'Hôpital's rule once, we notice that we have $\frac{\infty}{\infty}$ again. So we can repeat the L'Hôpital's rule and get the answer to our question:

$$\lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

5.1.2 Derivation from first principles

As the derivative at a point is the gradient, differentiation can be compared to finding gradients of lines: $m = \frac{y_2 - y_1}{x_2 - x_1}$.



Using the graph

$$x_1 = x \qquad x_2 = x + h$$
$$y_1 = f(x) \qquad y_2 = f(x+h)$$

Plugging into the equation of the gradient of a line

$$m = \frac{f(x+h) - f(x)}{x + h - x}$$

Taking the limit of h going to zero, such that the distance between the points becomes very small, one can approximate the gradient at a point of any function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5.2 Polynomials

As you have learnt in the section on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and thanks to it we can tell how steep the line will be. In fact gradients can be found for any function - the special thing about linear functions is that their gradient is always the same (given by m in $y = mx + c$). For polynomial functions the gradient is always changing. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of x.

Using the following steps, you can find the derivative function ($f'(x)$) for any polynomial function ($f(x)$).

Polynomial a mathematical expression or function that contains several terms often raised to different powers

e.g. $y = 3x^2$, $y = 121x^5 + 7x^3 + x$ or $y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$

Principles $y = f(x) = ax^n \quad \Rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nax^{n-1}.$

The (original) function is described by y or $f(x)$, the derivative (gradient) function is described by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$.

Derivative of a constant (number) 0

e.g. For $f(x) = 5$, $f'(x) = 0$

Derivative of a sum sum of derivatives.

If a function you are looking to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

e.g. $f(x) = ax^n$ and $g(x) = bx^m$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

5.2.1 Rules

DB 5.6

With more complicated functions, in which several functions are being multiplied or divided by one another (rather than just added or subtracted), you will need to use the product or quotient rules.

Product rule

When functions are *multiplied*: $y = uv$

then: $y' = uv' + u'v$

which is the same as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$

Quotient rule

When functions are *divided*: $y = \dfrac{u}{v}$

then: $y' = \dfrac{vu' - uv'}{v^2}$

which is the same as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Example.

Let $y = x^2 \cos x$, then

$$y' = x^2(\cos x)' + \left(x^2\right)' \cos x$$

$$= -x^2 \sin x + 2x \cos x$$

Let $y = \dfrac{x^2}{\cos x}$, then

$$y' = \frac{\left(x^2\right)' \cos x - x^2(\cos x)'}{(\cos x)^2}$$

$$= \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

Chain rule

When a function is inside another function: $y = g(u)$ where $u = f(x)$

then: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

A function inside another function is a composite function, $f \circ g(x)$, which we discussed in the Functions chapter

Differentiating with the chain rule

	Let $y = (\cos x)^2$, determine the derivative y'	
1.	Determine what the inside (u) and outside (y) functions are	Inside function: $u = \cos x$ Outside function: $y = u^2$
2.	Find u' and y'	$u' = \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x; \quad y' = \dfrac{\mathrm{d}y}{\mathrm{d}u} = 2u$
3.	Fill in chain rule formula	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$ $= 2u(-\sin x)$ $= -2\sin x \cos x$

5.3 Derivatives of alternative functions

Most common derivatives can be shown in the table below

$$\left(\sin(x)\right)' = \cos(x)$$

$$\left(\cos(x)\right)' = -\sin(x)$$

$$\left(\tan(x)\right)' = \sec^2(x)$$

$$\left(\arcsin(x)\right)' = \frac{1}{\sqrt{1-x^2}}$$

$$\left(\arccos(x)\right)' = -\frac{1}{\sqrt{1-x^2}}$$

$$\left(\arctan(x)\right)' = \frac{1}{1+x^2}$$

$$\left(\sec(x)\right)' = \sec(x)\cdot\tan(x)$$

$$\left(\csc(x)\right)' = -\csc(x)\cdot\cot(x)$$

$$\left(\cot(x)\right)' = -\csc^2(x)$$

$$(\mathrm{e}^x)' = \mathrm{e}^x$$

$$\left(\ln(x)\right)' = \frac{1}{x}$$

$$(a^x)' = a^x\left(\ln a\right)$$

$$\left(\log_a(x)\right)' = \frac{1}{x\ln(a)}$$

There is no need to learn these derivatives, since they all are in the data booklet. However, it is good to know how they work, especially for integration. Do not forget to use the chain rule when there is an inside function instead of just an x: Remember to multiply out the derivative of the inside function with the derivative of the function itself.

Example.

Find derivative of $\ln\left(x^3+2\right)$.

One way to do it, is to substitute inside as another letter. Let $u = x^3+2$. Then:

$$\left(\ln(u)\right)' = \frac{1}{u}\times u' = \frac{1}{x^3+2}\times\left(x^3+2\right)' = \frac{3x^2}{x^3+2}$$

5.3.1 Implicit differentiation

When we have a function that does not express y explicitly ($y =$) like in the previous methods, we must use *implicit differentiation*.

Steps to follow:

1. differentiate with respect to x, don't forget chain and x rules. Derivative of y is $\frac{\mathrm{d}y}{\mathrm{d}x}$
2. collect/gather terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$
3. solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

Implicit differentiation

Find the gradient at point $(0, 1)$ of $\mathrm{e}^{xy} + \ln\left(y^2\right) + \mathrm{e}^y = 1 + \mathrm{e}$

1.	Treat each part separately	e^{xy} becomes $y\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x} x\mathrm{e}^{xy}$ $\ln\left(y^2\right)$ becomes $2y \times \frac{1}{y^2} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y} \frac{\mathrm{d}y}{\mathrm{d}x}$ e^y becomes $\frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{e}^y$
2.	Collect/gather terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$	$y\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x} x\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x} \frac{2}{y} + \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{e}^y = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(x\mathrm{e}^{xy} + \frac{2}{y} + \mathrm{e}^y\right) = -y\mathrm{e}^{xy}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y\mathrm{e}^{xy}}{x\mathrm{e}^{xy} + \frac{2}{y} + \mathrm{e}^y}$
3.	Solve for the point $(0, 1)$	Substituting in $x = 0$ and $y = 1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{2 + \mathrm{e}}$

5.4 Tangent and normal equation

Tangent a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

Normal a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$

For any questions with tangent and/or normal lines, use the steps described in the following example.

Finding the linear function of the tangent.

Let $f(x) = x^3$. Find the equation of the tangent at $x = 2$

1.	Find the derivative and fill in value of x to determine slope of tangent	$f'(x) = 3x^2$ $f'(2) = 3 \cdot 2^2 = 12$
2.	Determine the y value	$f(x) = 2^3 = 8$
3.	Plug the slope m and the y value in $y = mx + c$	$8 = 12x + c$
4.	Fill in the value for x to find c	$8 = 12(2) + c \Rightarrow c = -16$ eq. of tangent: $y = 12x - 16$

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

Finding the linear function of the normal.

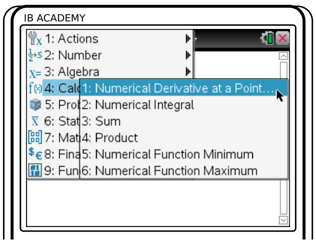
Let $f(x) = x^3$. Find the equation of the normal at $x = 2$

1.	______________________	$f'(2) = 12$
2.	______________________	$f(x) = 8$
3.	Determine the slope of the normal $m = \dfrac{-1}{\text{slope tangent}}$ and plug it and the y-value into $y = mx + c$	$m = \dfrac{-1}{12}$ $8 = -\dfrac{1}{12}x + c$
4.	Fill in the value for x to find c	$8 = -\dfrac{1}{12}(2) + c \Rightarrow c = \dfrac{49}{6}$ eq. of normal: $y = -\dfrac{1}{12}x + \dfrac{49}{6}$

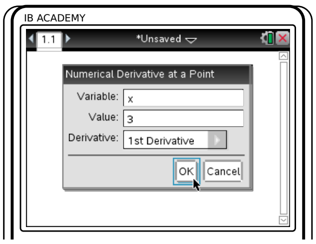
Steps 1, 2 and 4 are identical for the equation of the tangent and normal

To find the gradient of a function for any value of x.

$f(x) = 5x^3 - 2x^2 + x$. Find the gradient of $f(x)$ at $x = 3$.



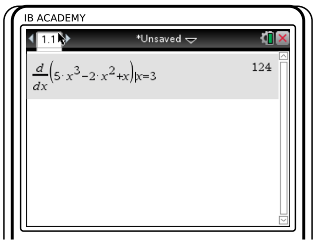
Press menu
`4: Calculus`
`1: Numerical Derivative at a Point`



Variable: x
Value: 3
Derivative: 1st Derivative

Enter the variable used in your function (x) and the value of x that you want to find. Keep the settings on `1st Derivative`

Press OK



$\frac{d}{dx}(5 \cdot x^3 - 2 \cdot x^2 + x)|x=3$ 124

Type in your function

press enter

In this case, $f'(3) = 124$

5.5 Turning points

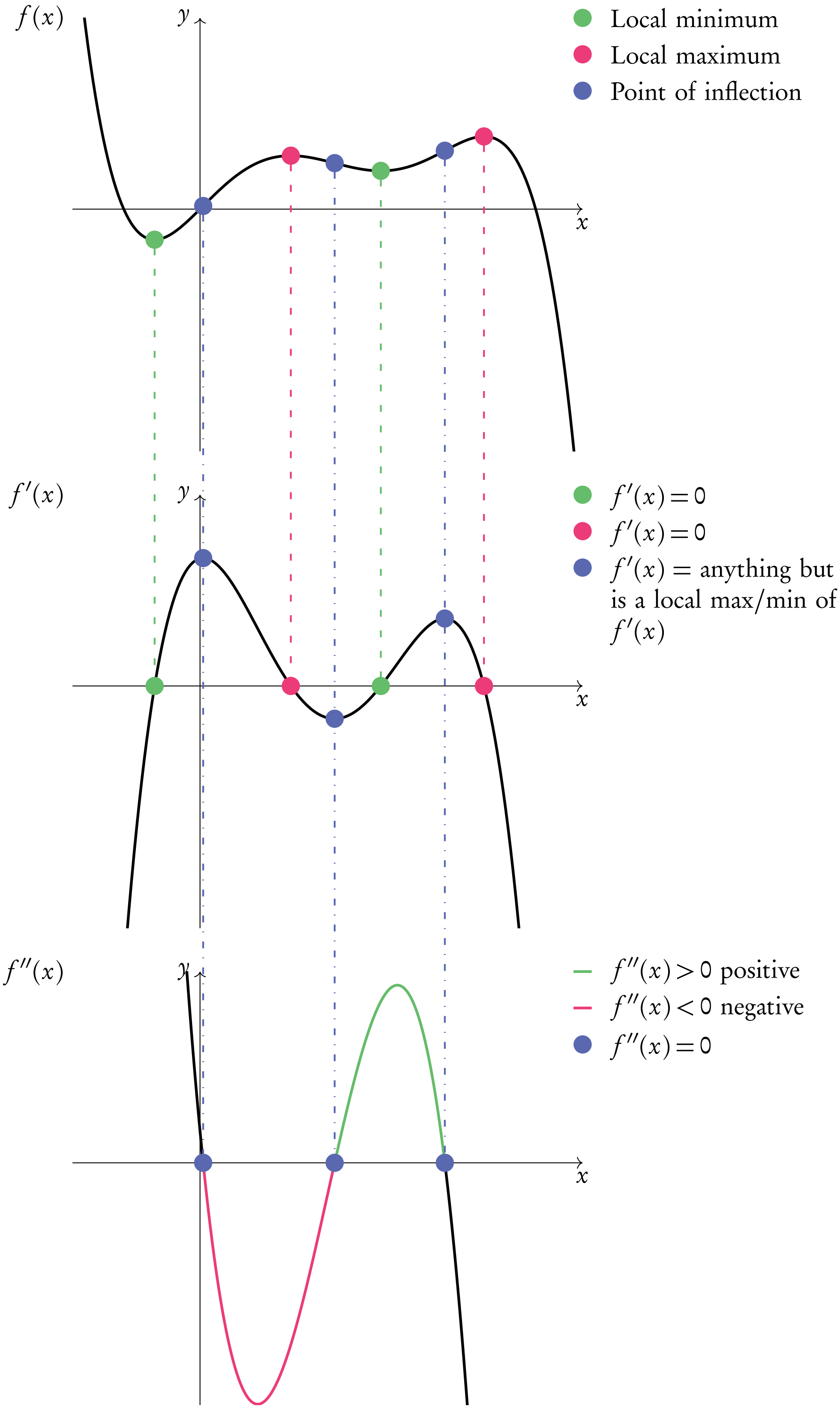
There are three types of turning points:

1. Local maxima
2. Local minima
3. Points of inflection

We know that when $f'(x) = 0$ there will be a maximum or a minimum. Whether it is a maximum or minimum should be evident from looking at the graph of the original function. If a graph is not available, we can find out by plugging in a slightly smaller and slightly larger value than the point in question into $f'(x)$. If the smaller value is negative and the larger value positive then it is a local minimum. If the smaller value is positive and the larger value negative then it is a local maximum.

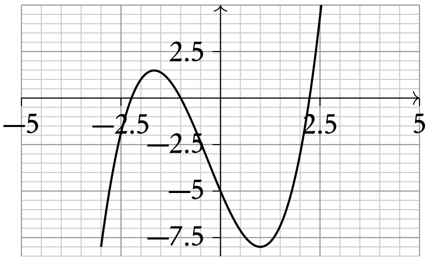
If you take the derivative of a derivative function (one you have already derived) you get the *second derivative*. In mathematical notation, the second derivative is written as y'', $f''(x)$ or $\frac{d^2y}{dx^2}$. We can use this to determine whether a point on a graph is a maximum, a minimum or a point of inflection as demonstrated in the following Figure 5.1.

Figure 5.1: Graph that shows a local maximum, a local minimum and points of inflection



Notice how the points of inflection of $f(x)$ are minima and maxima in $f'(x)$ and thus equal 0 in $f''(x)$

Finding turning points

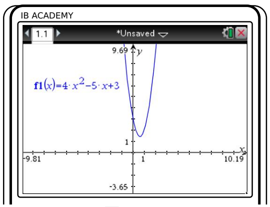


The function $f(x) = x^3 + x^2 - 5x - 5$ is shown. Use the first and second derivative to find its turning points: the minima, maxima and points of inflection (POI).

1.	Find the first and second derivative	$f'(x) = 3x^2 + 2x - 5$ $f''(x) = 6x + 2$
2.	Find x_{min} and x_{max} by setting $f'(x) = 0$	$3x^2 + 2x - 5 = 0$ GDC yields: $x = 1$ or $x = -\frac{5}{3}$
3.	Find y-coordinates by inserting the x-value(s) into the original $f(x)$	$f(1) = (1)^3 + (1)^2 - 5(1) - 5 = -8$, so x_{min} at $(1, -8)$. $f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2$ $-5\left(-\frac{5}{3}\right) - 5 = 1.48 (3 \text{ s.f.})$, so x_{max} at $\left(-\frac{5}{3}, 1.48\right)$.
4.	Find POI by setting $f''(x) = 0$	$6x + 2 = 0$
5.	Enter x-values into original function to find coordinates	$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - 5\left(-\frac{1}{3}\right) - 5$ $y = -3.26$ (3 s.f.) so POI at $\left(-\frac{1}{3}, -3.26\right)$

Finding turning points (local maximum/minimum) of a function using GDC

Find the coordinates of the local minimum for $f(x) = 4x^2 - 5x + 3$

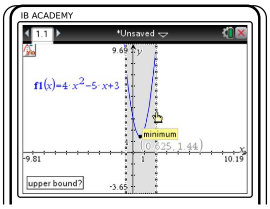


Press menu

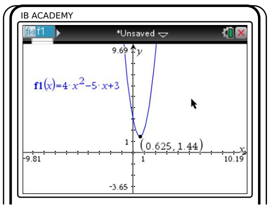
`6: Analyze graph`

or `2: Minimum`

or `3: Maximum`



Use the cursor to set the bounds (the min/max must be between the bounds)



So the coordinates of the minimum for $f(x)$ are $(0.625, 1.44)$

5.6 Sketching graphs

When sketching a graph, you will need the following information:

1. Intercepts,
2. Turning points (maximums, minimums and inflection points) and
3. Asymptotes

Sketching a function

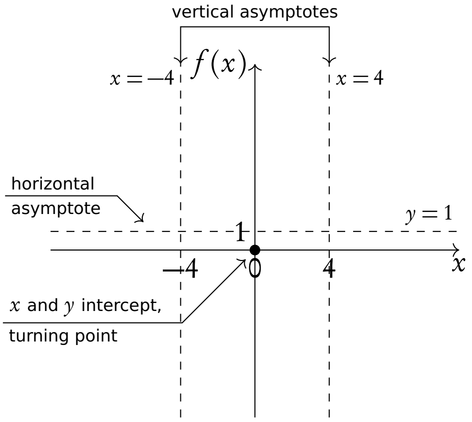
Sketch the function $f(x)=\frac{x^2}{x^2-16}$

1. Note down all information:

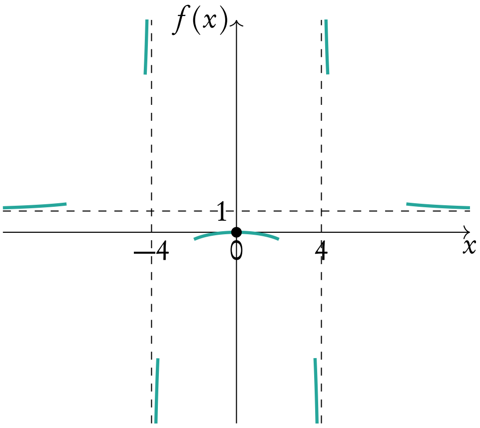
1. Intercepts: • y-intercept: $f(0)$ • x-intercept: $f(x)=0$	1. y-intercept when $x=0$: $f(0)=\frac{0^2}{0^2-16}=0 \quad (0,0)$ $f(x)=\frac{x^2}{x^2-16}=0 \quad x=0$ (same) This is the only x-intercept.
2. Turning points: • min/max: $f'(x)=0$ • inflection: $f''(x)=0$	2. Turning point: $f'(x)=\frac{-32x}{x^2-16^2}$, $x=0$ $(0,0)$ (Found with quotient rule). $f'=0$ when $x=0$.
3. Asymptotes: • vertical: denominator = 0, $x=-b$, for $\log(x+b)$ • horizontal: $\lim_{x\to\infty \text{ or } x\to-\infty}$, $y=c$, for a^x+c	3. Vertical asymptotes when $x^2-16=0$, so $x=4$ and $x=-4$. Horizontal asymptote: $\lim_{x\to\infty} f(x)=\frac{x^2}{x^2}=1$, so $y=1$
To find the y-coordinate, input the x-value into the original $f(x)$.	

2. Mark out information on axis

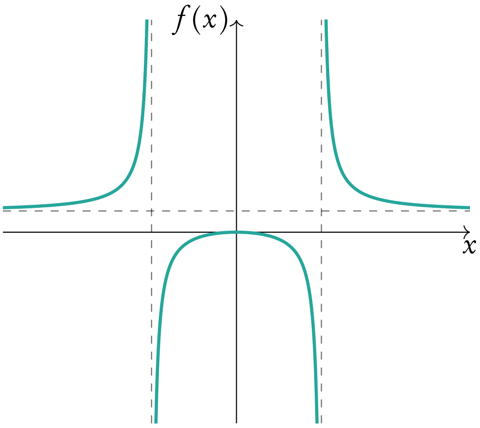
Clearly indicate them to guarantee marks



3. Think about where your lines are coming from



4. Join the dots



5.7 Applications

5.7.1 Kinematics

Kinematics deals with the movement of bodies over time. When you are given one function to calculate displacement, velocity or acceleration you can use differentiation or integration to determine the functions for the other two.

$$\begin{tikzcd}
\text{Displacement, } s \arrow[d, bend right, "\frac{\mathrm{d}s}{\mathrm{d}t}"'] \\
\text{Velocity, } v = \frac{\mathrm{d}s}{\mathrm{d}t} \arrow[u, bend right, "\int v\,\mathrm{d}t"'] \arrow[d, bend right, "\frac{\mathrm{d}v}{\mathrm{d}t}"'] \\
\text{Acceleration, } a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \arrow[u, bend right, "\int a\,\mathrm{d}t"']
\end{tikzcd}$$

The derivative represents the rate of change, i.e. the gradient of a graph. So, velocity is the rate of change in displacement and acceleration is the rate of change in velocity.

Answering kinematics questions.

A diver jumps from a platform at time $t = 0$ seconds. The distance of the diver above water level at time t is given by $s(t) = -4.9t^2 + 4.9t + 10$, where s is in metres. Find when velocity equals zero. Hence find the maximum height of the diver.

1.	Find an equation for velocity by differentiating equation for distance	$v(t) = -9.8t + 4.9$
2.	Solve for $v(t) = 0$	$-9.8t + 4.9 = 0, \quad t = 0.5$
3.	Put value into equation for distance to find height above water	$s(0.5) = -4.9(0.5)^2 + 4.9(0.5) + 10 =$ 11.225 m

5.7.2 Optimization

We can use differentiation to find minimum and maximum areas/volumes of various shapes. Often the key skill with these questions is to find an expression using simple geometric formulas and rearranging in order to differentiate.

Finding the minimum/maximum area or volume

The sum of height and base of a triangle is 40 cm. Find an expression for its area in terms of x, its base length. Hence find its maximum area.

1.	Find expressions for relevant dimensions of the shape	length of the base $(b) = x$ height + base = 40 so $h + x = 40$ area of triangle $A = \frac{1}{2}xh$
2.	Reduce the number of variables by solving the simultaneous equations	Since $h = 40 - x$, substitute h into A: $A = \frac{1}{2}x(40 - x) = -\frac{1}{2}x^2 + 20x$
3.	Differentiate	$f'(x) = -x + 20$
4.	Find x when $f'(x) = 0$	$-x + 20 = 0 \Rightarrow x = 20$
5.	Plug x value in $f(x)$	$-\frac{1}{2}20^2 + 20(20) = -200 + 400 = 200\,\text{cm}^2$

If an expression is given in the problem, skip to step 2 (e.g. cost/profit problems)

5.7.3 Related rates of change

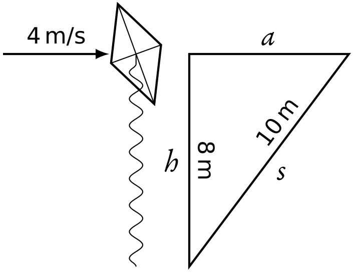
We can also use differentiation to find how fast a specific variable changes, dependent on the other ones. The key to getting good at related rates is to learn how to setup the equation and then how to differentiate with respect to time or a specific variable.

Finding the related rate of change

A flying kite is flying at a constant 8 m height above the ground. It moves away from us horizontally with a constant speed of 4 m/s. How fast does the string run out measured in m/s, when the kite is 10 m away from us?

1. Make a sketch

h (height), s (string length), a (distance from initial position in horizontal direction)



2. Identify each variable and what to find.

$\frac{dh}{dt} = 0$ $\quad \frac{ds}{dt} = ?$ $\quad \frac{da}{dt} = 4$

$h = 8$ $\quad s = 10$ $\quad a = 6$

3. Setup an equation. Pythagoras is the most obvious setup.

$$h^2 + a^2 = s^2$$

4. Differentiate with respect to time (dt)

$$2h\frac{dh}{dt} + 2a\frac{da}{dt} = 2s\frac{ds}{dt}$$

5. Finally, solve for the required variable

Here $\frac{ds}{dt}$:

$$\frac{ds}{dt} = \left(2h\frac{dh}{dt} + 2a\frac{da}{dt}\right) \div (2s)$$

$$\frac{ds}{dt} = 2.4\,\mathrm{m/s}$$

5.8 Maclaurin series

Maclaurin series, also known as Taylor series in a general case, are a way to approximate a function with polynomials. The series is calculated by finding n-th derivatives of a function at a point $x = 0$. The more terms you have in the series, the more accurate is the series.

Maclaurin series of degree n is an expansion of a function $f(x)$ around point $x = 0$

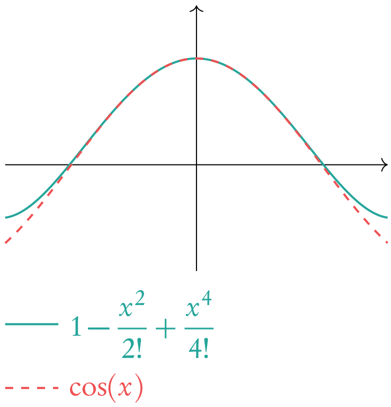
$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n$$

Example.

Calculate the 4th order Maclaurin series of $\cos(x)$.

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f''(x) &= -\cos(x) & f''(0) &= -1 \\ f'''(x) &= -\sin(x) & f'''(0) &= 0 \\ f''''(x) &= \cos(x) & f''''(0) &= 1 \end{aligned}$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$



Maclaurin series for special functions one can derive Maclaurin series for most used functions to reuse them in more complicated problems

DB 5.19

$$\mathrm{e}^{x}=1+x+\frac{x^{2}}{2}+\dots$$

$$\ln(1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\dots$$

$$\sin(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\dots$$

$$\cos(x)=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\dots$$

$$\arctan(x)=x-\frac{x^{3}}{3}+\frac{x^{5}}{5}-\dots$$

Example.

Find Maclaurin series of e^{x^2}. We can substitute x^2 for x in e^x Maclaurin series to get the answer quickly.

$$\mathrm{e}^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}$$

$$\mathrm{e}^{x^{2}}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

6

INTEGRATION

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6.1. Indefinite integral 100

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Integration with an internal function

$$\int f(ax+b)\, dx$$

Integrate normally and multiply by $\dfrac{1}{\text{coefficient of } x}$

Integration by substitution

$$\int f\left(g(x)\right) \cdot g'(x)\, dx$$

6.2. Definite integral 106

$$\int_a^b f(x)\, dx = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\, dx$$

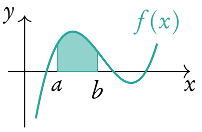
Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

Area between a curve and the x-axis

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\, dx$$

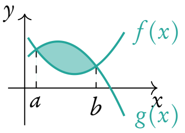


Note: the area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis.

Area between two curves

Using definite integrals you can also find the areas enclosed between curves.

$$A_{\text{between}} = \int_a^b \left(g(x) - f(x)\right) dx$$



With $g(x)$ as the "top" function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

Volume of revolution

$$V = \pi \int_a^b y^2 \, dx = \int_a^b \pi y^2 \, dx$$

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution.

6.1 Indefinite integral and boundary condition

Integration is essentially the opposite of derivation. The following equation shows how to integrate a function:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

As you can see, every time you integrate the power on your variable will increase by 1 and you divide by the new power. This is opposite of what happens when you derive, then the power always decreases and you multiply by the original power.

Whenever you integrate you also **always add** $+C$ to this function. This accounts for any constant that may have been lost while deriving. As you may have noticed, whenever you do derivation any constants that were in the original function, $f(x)$, become 0 in the derivative function, $f'(x)$. In order to determine the value of C, you need to fill in a point that lies on the curve to set up an equation in which you can solve for C.

This is the same thing you need to do when finding the y-intercept, C, for a linear function – see Functions: Linear functions.

Standard integration

Let $f'(x) = 12x^2 - 2$
Given that $f(-1) = 1$, find $f(x)$.

1.	Separate summed parts (optional)	$\int 12x^2 - 2\,\mathrm{d}x = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x$
2.	Integrate	$f(x) = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x =$ $\frac{12}{3}x^3 - 2x + C$
3.	Fill in values of x and $f(x)$ to find C	Since $f(-1) = 1$, $4(-1)^3 - 2(-1) + C = 1$ $C = 3$ So: $f(x) = 4x^3 - 2x + 3$

6.1.1 Integration with an internal function

$$\int f(ax+b)\,\mathrm{d}x \qquad \text{integrate normally and multiply by } \frac{1}{\text{coefficient of } x}$$

Example.

Find the following integrals:

$\int e^{3x-4}\,\mathrm{d}x$	$\int \cos(5x-2)\,\mathrm{d}x$
Coefficient of $x = 3$, so	Coefficient of $x = 5$, so
$\int e^{3x-4}\,\mathrm{d}x = \frac{1}{3}e^{3x-4} + C$	$\int \cos(5x-2)\,\mathrm{d}x = \frac{1}{5}\sin(5x-2) + C$

6.1.2 Integration by substitution

$$\int f\left(g(x)\right)\cdot g'(x)\,\mathrm{d}x$$

Integration by substitution questions are recognisable by a function and its derivative inside the function. Learning to spot these quickly is a matter of practice. Once you have identified the inside functions, the rest is fairly straight forward.

Integrate by substitution

	Find $\int 3x^2 e^{x^3}\,\mathrm{d}x$	
1.	Identify the inside function u, this is the function whose derivative is also inside $f(x)$	$g(x) = u = x^3$
2.	Find the derivative $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$	$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$
3.	Substitute u and $\frac{\mathrm{d}u}{\mathrm{d}x}$ into the integral (this way $\mathrm{d}x$ cancels out)	$\int e^u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + C$
4.	Substitute u back to get a function with x	$\int e^u + C = e^{x^3} + C$

6.1.3 Integration by parts

DB

General statement:

$$\int u \cdot \mathrm{d}v = uv - \int v \cdot \mathrm{d}u \qquad \text{or} \qquad \int f(x)g'(x)\,\mathrm{d}x = fg - \int f'g\,\mathrm{d}x$$

Example.

Solve $\int x \sin x \, \mathrm{d}x$

If $f(x) = x$ then $f'(x) = 1$ and the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$

$$-x\cos x - \int 1 \cdot (-\cos x)\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C$$

Sometimes it may be necessary to do integration by parts multiple times.

Example.

Solve $\int \mathrm{e}^{2x} \sin x \, \mathrm{d}x$

We know that $\sin x$ is the derivative of $\cos x$

$$u = \sin x \qquad \mathrm{d}u = \cos x\,\mathrm{d}x$$

$$\mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x \qquad v = \frac{\mathrm{e}^{2x}}{2}$$

Using the formula given in the data booklet and the information above:

$$\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x = \frac{\mathrm{e}^{2x}\sin x}{2} - \int \frac{\mathrm{e}^{2x}}{2}\cos x\,\mathrm{d}x$$

Unfortunately this is still not nice to solve so we need to repeat the procedure along the same line of reasoning for the integral $\int \frac{\mathrm{e}^{2x}}{2}\cos x\,\mathrm{d}x$.

We know trigonometric functions recur – taking the integral twice would bring us back to the same trigonometric identity, apart from containing the opposite sign.

$$u = \cos x \qquad \mathrm{d}u = -\sin x\,\mathrm{d}x$$

$$\mathrm{d}v = \frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x \qquad v = \frac{\mathrm{e}^{2x}}{4}$$

We again construct the formula of integration by parts.

$$\int \frac{\mathrm{e}^{2x}}{2}\cos x\,\mathrm{d}x = \cos x\frac{\mathrm{e}^{2x}}{4} - \int \frac{\mathrm{e}^{2x}}{4}\sin x\,\mathrm{d}x$$

Example.

Now we want to combine both equations to solve for the original integral.

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \cos x\frac{e^{2x}}{4} + \int \frac{e^{2x}}{4}\sin x\,dx$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \cos x\frac{e^{2x}}{4}$$

$$= \frac{2}{5}\sin x e^{2x} - \frac{1}{5}\cos x e^{2x}$$

6.1.4 Special ways of integration

Some integrals cannot be solved easily by using any of the usual methods. Thus there exist some special tricks to help solve such integrals that are good to remember.

Two of the common "hard" integrals are $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$. It might look easy, but neither substitution or integration by parts work here. Thus, it is worth remembering the double identity formula:

$$\cos(2\vartheta) = 2\cos^2(\vartheta) - 1 = 1 - 2\sin^2(\vartheta)$$

By solving for $\sin^2(\vartheta)$ or $\cos^2(\vartheta)$, we get these two formulas:

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

By substituting $\sin^2(x)$ or $\cos^2(x)$ with corresponding formulas, it is possible to solve the integrals very fast.

There are also three other useful substitutions, when you meet either of the following equations:

$$\begin{aligned} a^2 - x^2, &\quad x = a\sin(\vartheta) \\ x^2 - a^2, &\quad x = a\cos(\vartheta) \\ x^2 + a^2, &\quad x = a\tan(\vartheta) \end{aligned}$$

Where a is a real number and ϑ is variable that we substitute x for. Usually those equations should be substituted when they are located in the square root, fraction or both.

Solve integral

Solve the integral $\int_0^{\frac{5}{2}} \sqrt{25-4x^2}\,dx$.

1. Recognise which substitution works.

Here it is a^2-x^2.

$$2x=5\sin(\vartheta)$$

$$dx=\frac{5}{2}\cos(\vartheta)\,d\vartheta$$

2. Convert boundaries on the integral.

$$\vartheta=\arcsin\left(\frac{2}{5}x\right)$$

$$\vartheta=\arcsin(0)=0$$

$$\vartheta=\arcsin\left(\frac{2}{5}\times\frac{5}{2}\right)=\frac{\pi}{2}$$

3. Do the substitution and solve the integral.
Use Pythagorean identities and double angle identities, when necessary.

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \sqrt{25-25\sin^2(\vartheta)}\times\frac{5}{2}\cos(\vartheta)\,d\vartheta\\
&=\int_0^{\frac{\pi}{2}} 5\sqrt{1-\sin^2(\vartheta)}\times\frac{5}{2}\cos(\vartheta)\,d\vartheta\\
&=\int_0^{\frac{\pi}{2}} 5\cos(\vartheta)\times\frac{5}{2}\cos(\vartheta)\,d\vartheta\\
&=\int_0^{\frac{\pi}{2}} \frac{25}{2}\cos^2(\vartheta)\,d\vartheta\\
&=\int_0^{\frac{\pi}{2}} \frac{25}{2}\left(\frac{1+\cos(2\vartheta)}{2}\right)d\vartheta\\
&=\frac{25}{2}\left(\frac{1}{2}\vartheta+\frac{1}{4}\sin(2\vartheta)\right)\Bigg|_0^{\frac{\pi}{2}}\\
&=\frac{25}{2}\left(\frac{1}{2}\times\frac{\pi}{2}+\frac{1}{4}\sin\left(2\times\frac{\pi}{2}\right)\right.\\
&\qquad\left.-\frac{1}{2}\times 0-\frac{1}{4}\sin(0)\right)\\
&=\frac{25}{8}\pi
\end{aligned}$$

6.1.5 Using partial fractions

As you have learnt way back in algebra topic, one can use partial fractions to split a bigger fraction into smaller ones. There are a few uses for that method including integration. Let's see how we can use partial fractions here.

Example.

Find the integral of $\int \frac{1}{x^2+3x+2}\,\mathrm{d}x$

Let's first express the integral as partial fractions

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

Now it is way easier to find the integral of individual terms

$$\begin{aligned}\int \frac{1}{x^2+3x+2}\,\mathrm{d}x &= \int \frac{1}{x+1} - \frac{1}{x+2}\,\mathrm{d}x \\ &= \ln|x+1| - \ln|x+2| + C \\ &= \ln\left|\frac{x+1}{x+2}\right| + C\end{aligned}$$

6.2 Definite integral

If there are limit values indicated on your integral, you are looking to find a definite integral. This means that these values will be used to find a numeric answer rather than a function.

This is done in the following way, where the values for a and b are substituted as x-values into your indefinite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

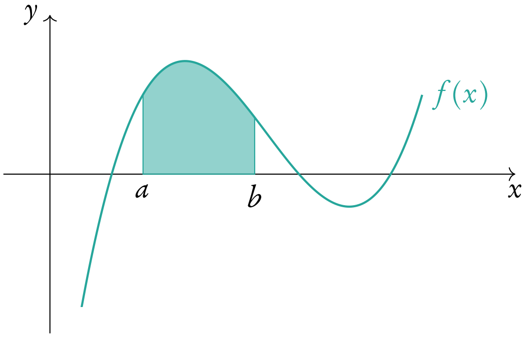
Solving definite integrals

Find $\int_3^7 12x^2 - 2\,\mathrm{d}x$, knowing that $F(x) = 4x^3 - 2x$

1.	Find the indefinite integral (without $+C$)	$\int_3^7 12x^2 - 2\,\mathrm{d}x = \left[4x^3 - 2x\right]_3^7$
2.	Fill in: $F(b) - F(a)$ (integral $x = b$) $-$ (integral $x = a$)	$= \left[4(7)^3 - 2(7)\right] - \left[4(3)^3 - 2(3)\right]$ $= 1256$

6.2.1 Area

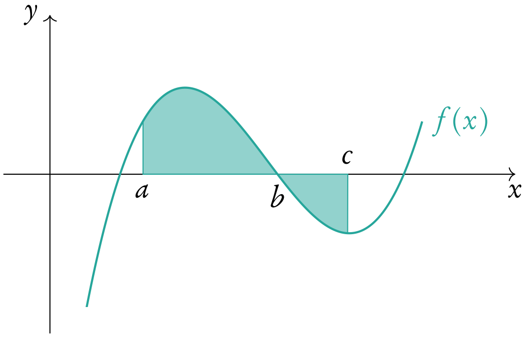
Area between a curve and the x-axis



DB 5.5

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$



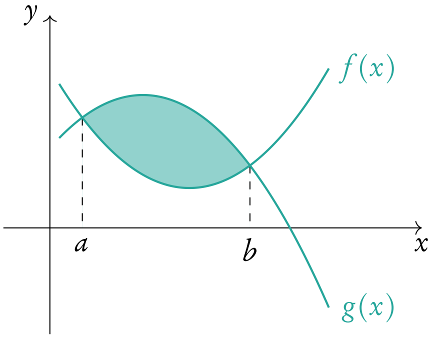
The area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis. So

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x + \left|\int_b^c f(x)\,\mathrm{d}x\right|$$

or

$$A_{\text{curve}} = \int_a^c |f(x)|\,\mathrm{d}x$$

Area between two curves



Using definite integrals you can also find the areas enclosed between curves:

$$A_{\text{between}} = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x$$

With $g(x)$ as the “top” function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

Finding areas with definite integrals.

Let $y = x^3 - 4x^2 + 3x$

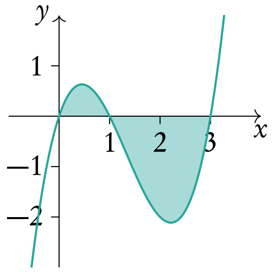
Find the area from $x = 0$ to $x = 3$.

1. Find the x-intercepts: $f(x) = 0$

$x^3 - 4x^2 + 3x = 0$, using the GDC:

$x = 0$ or $x = 1$ or $x = 3$

2. If any of the x-intercepts lie within the range, sketch the function to see which parts lie above and below the x-axis.



3. Setup integrals and integrate

Left: $\displaystyle\int_0^1 x^3 - 4x^2 + 3x \, \mathrm{d}x =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0)$$

$$= \frac{5}{12}$$

Right: $\displaystyle\int_1^3 x^3 - 4x^2 + 3x \, \mathrm{d}x =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

$$= \left(\frac{1}{4}(3)^4 - \frac{4}{3}(3)^3 + \frac{3}{2}(3)^2\right)$$

$$- \left(\frac{1}{4}(1)^4 - \frac{4}{3}(1)^3 + \frac{3}{2}(1)^2\right)$$

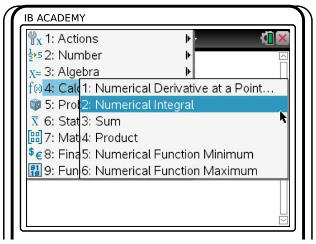
$$= -\frac{8}{3}$$

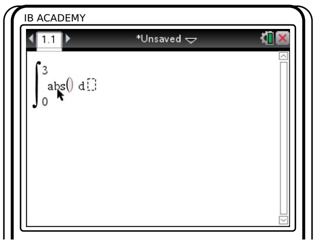
4. Add up the areas (and remember areas are never negative!)

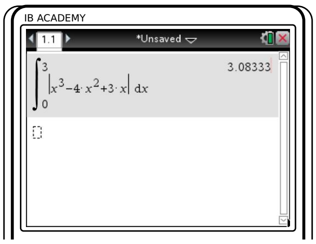
$$\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

Alternatively, use the calculator to find areas

Calculate the area between $\int_0^3 x^3 - 4x^2 + 3x$ and the x-axis







$\int_0^3 |x^3-4\cdot x^2+3\cdot x|\,dx$ 3.08333

Press menu
`4: Calculus`
`2: Numerical integral`

Enter the boundaries and before putting the function.

Press ∞β°, choose ‘`abs(`’

Enter the function and place the variable (usually x) after d

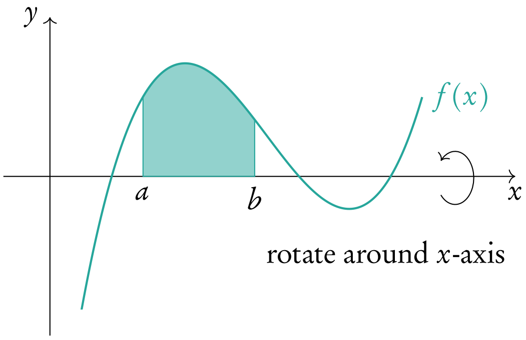
In this case, the area is 3.083

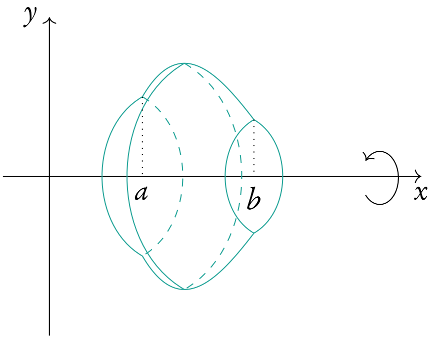
6.2.2 Volume of revolution

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution.

DB 6.5

$$V = \pi \int_a^b y^2\,\mathrm{d}x \quad \equiv \quad V = \int_a^b \pi y^2\,\mathrm{d}x$$





Example.

Find the area from $x = 1$ to $x = 4$ for the function $y = \sqrt{x}$.

$$A = \int_1^4 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right] = \frac{14}{3}$$

This area is rotated 360° $(= 2\pi)$ around the x-axis. Find the volume of the solid.

$$V = \pi\int_1^4 \sqrt{x}^2\,\mathrm{d}x = \pi\int_1^4 x\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_1^4 = \pi\left(\left[\frac{1}{2}(4)^2\right] - \left[\frac{1}{2}(1)^2\right]\right) = \frac{15\pi}{2}$$

It is also possible to find the volume of revolution around the y-axis. It requires some additional steps, but in general it is not much different from the volume of revolution around the x-axis.

Find the volume of revolution

Find the volume of revolution of function $y = x^2$ from $x = 1$ to $x = 3$ around y-axis.

1.	Convert function to $x =$ formula	$y = x^2$	$x = \sqrt{y}$
2.	Convert x coordinates to y coordinates.	$y(1) = 1^2 = 1$ $y(3) = 3^2 = 9$	
3.	Integrate the function with respect to $\mathrm{d}y$	$\pi\int_1^9 \sqrt{y}^2\,\mathrm{d}y = \pi\int_1^9 y\,\mathrm{d}y$ $= \pi\left[\frac{1}{2}y^2\right]\Big\vert_1^9$ $= \pi\left(\frac{1}{2}\times 81 - \frac{1}{2}\times 1\right) = 40\pi$	

To find volume of revolution between two graphs, use the following formula (works the same way with $\mathrm{d}y$):

$$V = \int_a^b \pi\left[(\text{Outer radius})^2 - (\text{Inner radius})^2\right]\mathrm{d}x$$

6.3 Ordinary differential equations

An ordinary differential equation (ODE) is an equation that relates functions of an independent variable and its derivatives. Differential equations are really important in mathematics as they allow to solve a lot of real world and science problems. There are several methods for solving the differential equations, including analytical approaches and numerical methods.

6.3.1 Separation of variables

One of the common approaches to solve an ODE is to use method of separation of variables. It can be used when one is able to separate all y dependent terms to the left and all x dependent terms to the right. Then it is possible to solve each side in relation to their own variables.

Solving an ODE using separation of variables.

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$ which satisfies the initial condition $y(0) = 2$.

1.	Identify that the ODE can be solved using separation of variables and move all y and x dependent terms to their respective sides.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$ $\frac{\mathrm{d}y}{y} = 2x\,\mathrm{d}x$
2.	Integrate each side separately. Remember to add a constant on the x side.	$\int \frac{\mathrm{d}y}{y} = \int 2x\,\mathrm{d}x$ $\ln(y) = x^2 + C$
3.	Solve for y.	$y = \mathrm{e}^{x^2+C} = k\mathrm{e}^{x^2}$
4.	Plug in initial conditions to find the value of the constant.	$y(0) = 2 = k\mathrm{e}^0 = k$ $y(x) = 2\mathrm{e}^{x^2}$

6.3.2 Using substitution $y = vx$

Sometimes an ODE cannot be solved using separation of variables because the terms cannot be separated. Then we have to use other techniques. One of these techniques requires to use substitution $y = vx$. It is good to use it when $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written as $f\left(\frac{y}{x}\right)$.

Solving an ODE using substitution.

	Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = 1$.	
1.	Make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject.	$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = 1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 1$
2.	After unsuccessfully trying to separate the variables, use substitution $y = vx$.	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{vx}{x} + 1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = v + 1$
3.	Find $\frac{\mathrm{d}v}{\mathrm{d}x}$ using implicit differentiation.	$y = vx$ $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
4.	Substitute for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the original equation and solve the ODE.	$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + 1$ $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$ $\int \mathrm{d}v = \int \frac{\mathrm{d}x}{x}$ $v = \ln(x) + C = \ln(kx)$
5.	Substitute back using $v = \frac{y}{x}$	$\frac{y}{x} = \ln(kx)$ $y = x\ln(kx)$

6.3.3 Using integrating factor

A few other differential equations can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)$. To solve this type of differential equation we need to multiply both sides by an integrating factor $I=\mathrm{e}^{\int P(x)\mathrm{d}x}$. In that case one can use a reverse product rule on the left hand side and thus can easily integrate the right hand side directly to solve the ODE.

Solving an ODE using integrating factor.

Solve $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3y}{x}=\frac{e^x}{x^3}$.

1.	Identify that the ODE can be solved using the integrating factor and identify P(x).	$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3y}{x}=\frac{e^x}{x^3}$ $P(x)=\frac{3}{x}$
2.	Calculate the integrating factor I.	$I=\mathrm{e}^{\int P(x)\mathrm{d}x}=\mathrm{e}^{\int \frac{3}{x}\mathrm{d}x}$ $I=\mathrm{e}^{3\ln(x)}=x^3$
3.	Multiply the ODE by the integrating factor I and apply reverse product rule on the left hand side.	$\frac{\mathrm{d}y}{\mathrm{d}x}x^3+\frac{3yx^3}{x}=\frac{x^3e^x}{x^3}$ $\frac{\mathrm{d}y}{\mathrm{d}x}x^3+3yx^2=e^x$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(yx^3\right)=e^x$
4.	Integrate the right hand side and solve for y.	$yx^3=\int e^x dx=e^x+C$ $y=\frac{e^x+C}{x^3}$

6.3.4 Euler's method

There are a lot of differential equations that are hard or impossible to solve analytically. Yet we still would like to know the value of a function given its initial conditions. Then we have to use numerical methods to solve the differential equation. Euler's method is a simple method that allows to find a value of a function given its first derivative and initial conditions. It is easy to derive the Euler's method from the approximation of a derivative at a point:

$$y'(x) = \frac{y(x+h) - y(x)}{h}$$

Solving for $y(x+h)$ gives us:

$$y(x+h) = y(x) + y'(x)h$$

Where we can rename for recursive purposes $y(x+h) = y_{n+1}$, $y(x) = y_n$ and $y'(x) = f(x_n, y_n)$, where $f(x_n, y_n)$ is our differential equation:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Now knowing initial conditions and using a small step h, it is possible to find an approximate value of y at required point x. The smaller step h, the more accurate is the solution. But it also takes longer to calculate it. It is recommended to create a table with all required variables to keep it clean and orderly.

Example.

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem $\frac{dy}{dx} = \sin(x+y)$, $y(0) = 1$ at $y(0.5)$

n	x_n	y_n	$f(x_n, y_n)$
0	0.0	1	0.8415...
1	0.1	$1.0000 + 0.1 \times 0.8415 = 1.0842$	0.9262...
2	0.2	$1.0842 + 0.1 \times 0.9262 = 1.1768$	0.9812...
3	0.3	$1.1768 + 0.1 \times 0.9812 = 1.2749$	1.0000...
4	0.4	$1.2749 + 0.1 \times 1.0000 = 1.3749$	0.9792...
5	0.5	$1.3749 + 0.1 \times 0.9792 = 1.4728$	

Thus $y(0.5) \approx 1.47$. A computer gives numerical solution equal to $y(0.5) = 1.47825\ldots$ So we are about 0.01 off the actual answer. That is pretty good for such a simple numerical method!

PROBABILITY

7

Table of contents & cheatsheet

Definitions

Sample space the list of all possible outcomes.

Event the outcomes that meet the requirement.

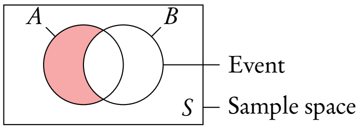
Probability for event A, $P(A)=\dfrac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$.

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

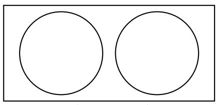
Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.

Conditional probability the probability of A, given that B has happened: $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$.

7.1. Single events 116



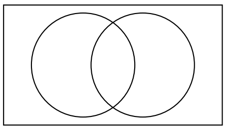
Mutually exclusive

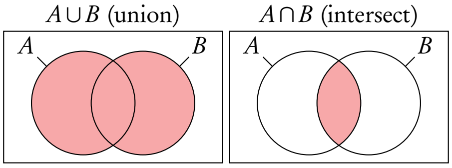


$$P(A\cup B)=P(A)+P(B)$$
$$P(A\cap B)=0$$

Combined events

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$P(A\cap B)=P(A)+P(B)-P(A\cup B)$$





If independent: $P(A\cap B)=P(A)\times P(B)$.

Compliment, A' where $P(A')=1-P(A)$

Exhaustive when everything in the sample space is contained in the events

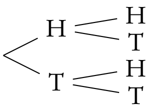
7.2. Multiple events 118

Probabilities for successive events can be expressed through tree diagrams or a table of outcomes.

Table of outcomes

	H	T
H	H,H	H,T
T	T,H	T,T

Tree diagram



In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

7.3. Distributions 121

For a distribution by function the domain of X must be defined as $\sum P(X=x)=1$.

Expected value $E(X)=\sum xP(X=x)$

Binomial distribution $X\sim B(n,p)$ used in situations with only 2 possible outcomes and lots of trials

$$P(X=x)=\binom{n}{r}p^{r}(1-p)^{n-r}$$

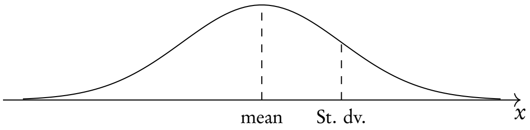
where $\binom{n}{r}=\text{nCr}=\dfrac{n!}{r!(n-r)!}$, $n=$ number of trials, $p=$ probability of success, $r=$ number of success.

On calculator:

- Binompdf(n,p,r) $P(X=r)$
- Binomcdf(n,p,r) $P(x\leq r)$

Mean $=np$
Variance $=npq$

Normal distribution $X\sim N(\mu,\sigma^2)$



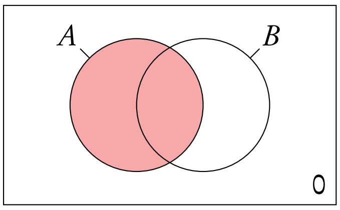
where μ =mean, σ =standard deviation

On calculator:

- normcdf(lower bound, upper bound, $=\mu$, σ)
- invnorm(area, $=\mu$, σ)

7.1 Single events (Venn diagrams)

Probability for single events can be visually expressed through Venn diagra



Sample space the list of all possible outcomes.

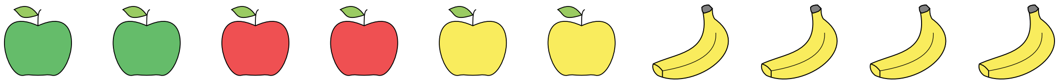
Event the outcomes that meet the requirement.

Probability for event A,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

Here the shaded circle.

Imagine I have a fruit bowl containing 10 pieces of fruit: 6 apples and 4 bananas.



I pick a piece of fruit. Below are some common situations with Venn diagrams.

These events are also exhaustive as there is nothing outside of the events (nothing in the sample space).

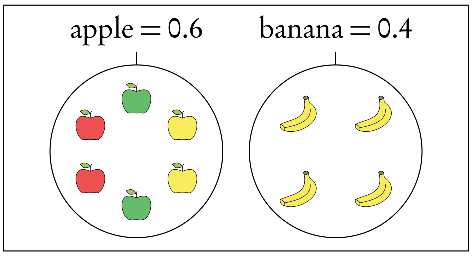
Mutually exclusive

Example.

What is the probability of picking each fruit?

Events do not overlap

$$P(A \cup B) = P(A) + P(B)$$
$$P(A \cap B) = 0$$



$$P(A) = \frac{6 \text{ Apples}}{10 \text{ pieces of fruit}} = 0.6$$
$$P(B) = \frac{4 \text{ Bananas}}{10 \text{ pieces of fruit}} = 0.4$$

In independent events $P(A \cap B) = P(A) \times P(B)$. It will often be stated in questions if events are independent.

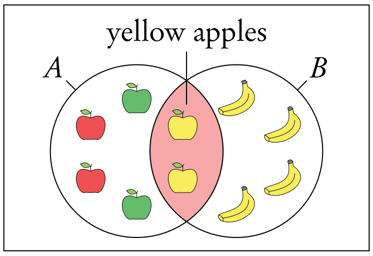
Combined events

Example.

Of the apples 2 are red, 2 are green and 2 are yellow.
What is the probability of picking a yellow apple?

The intersect is the area the events overlap.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



A: apples

B: yellow fruit

$$P(\text{yellow apple}) = \frac{\text{2 apples}}{\text{10 pieces of fruit}} = 0.2$$

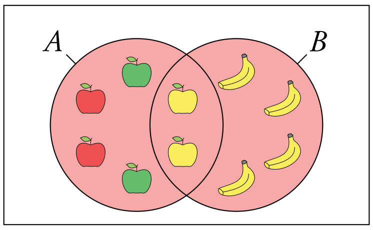
Example.

What is the probability of picking an apple or a yellow fruit?

The union is the area contain by both events.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When an event is exhaustive the probability of the union is 1.



A: apples

B: yellow fruit

Event is exhaustive so probability of union is 1.

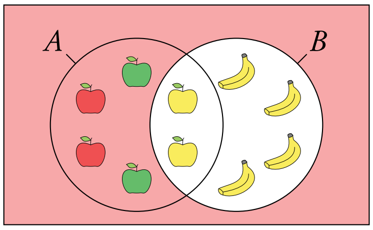
Compliment

Example.

What is the probability of not picking a yellow fruit?

Everything that is not in the stated event.

$$P(A') = 1 - P(A)$$



A: apples

B: yellow fruit

$P(B') = 1 - P(B) = 1 - 0.6 = 0.4$

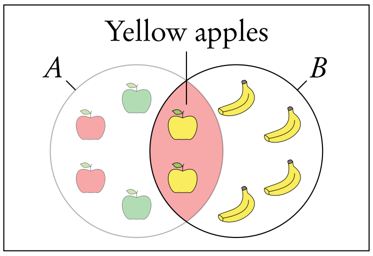
Conditional

Example.

What is the probability of picking an apple given I pick a yellow fruit?

The probability given that some condition is already in place.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

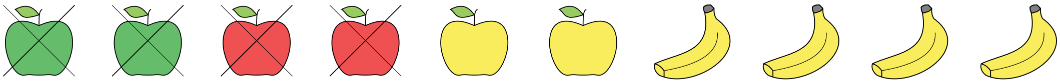


A: apples

B: yellow fruit

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{(0.2+0.4)} = \frac{1}{3}$$

You can think of this as using B as the sample space, or removing the non yellow apples from the fruit bowl before choosing.



7.2 Multiple events (tree Diagrams)

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.

Conditional probability the probability of A, given that B has happened:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Questions involving dependent events will often involve elements that are drawn “without replacement”. Remember that the probabilities will be changing with each new set of branches.

Probabilities for successive events can be expressed through tree diagrams or a table of outcomes. Often at standard level you will deal with two successive events, but both methods can be used for more. In general, if you are dealing with a question that asks for the probability of:

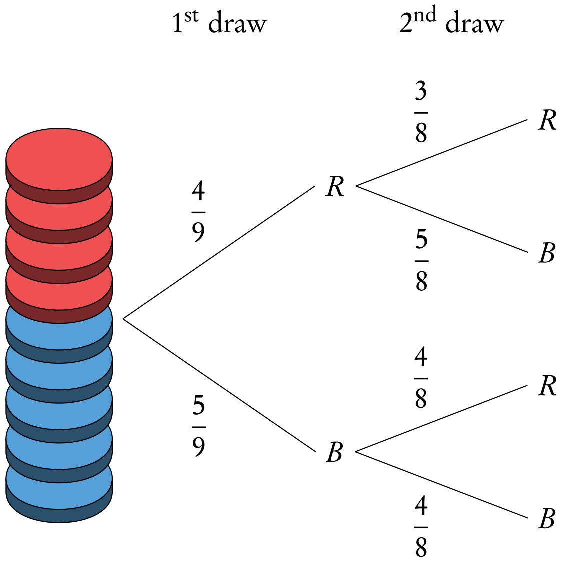
- one event **and** another, you **multiply**
- one event **or** another, you **add**

Tree diagrams

Example.

Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red $\left(\frac{4}{9} \times \frac{3}{8}\right)$.



R and B: $\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

R and B: $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

B and R: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

B and R: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).

What is the probability to draw one red and one blue disk?

P(one red **and** one blue)

$(P(R)$ and $P(B))$ **or** $(P(B)$ and $P(R))$

$(P(R) \times P(B))$ $(P(B) \times P(R))$

$$\frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

It is common for conditional probability questions to relate to previous answers.

What is the probability to draw at least one red disk?

P(**at least** one red)

$P(R$ and $R) + P(B$ and $R) + P(R$ and $B) = 1 - P(B$ and $B)$

$$\frac{12}{72} + \frac{20}{72} + \frac{20}{72} = 1 - \frac{20}{72} = \frac{52}{72} = \frac{13}{18}$$

What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk}) = \frac{P(\text{one red disk and one blue disk})}{P(\text{at least one red disk})} = \frac{\frac{5}{9}}{\frac{13}{18}} = \frac{10}{13}$$

Tables of Outcomes

A table of outcomes has the possible outcomes for one event in the first row and the possible outcomes for another event it the first column. The table is then filled in with either the combination of these outcomes or the number of items (or probability) that fall into both events.

Example.

Table of outcomes for two flips of a fair coin

	H	T
H	H,H	H,T
T	T,H	T,T

Example.

Table of outcomes for three machines and the average number of defective and non-defective items they make.

	Defective	Non-defective
Machine I	6	120
Machine II	4	80
Machine III	10	150

7.3 Distributions

Probability distribution a list of each possible value and their respective probabilities.

Expected value $E(X) = \sum xP(X = x)$

DB 4.7

We can take any of the examples above and create a probability distribution from them. It is important to define the factor X for which the probability applies. Once tabulated we can use the distribution to find the expected value. It is best to think of this as the average value you would get if you repeated the action many times.

Probability distributions

A fair coin is tossed twice, X is the number of heads obtained.

1. Draw a sample space diagram

	H	T
H	H, H	H, T
T	T, H	T, T

2. Tabulate the probability distribution

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(The sum of $P(X = x)$ always equals 1)

3. Find the expected value of X: $E(X)$

$$E(X) = \sum xP(X = x)$$
$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

So if you toss a coin twice, you expect to get heads once.

7.3.1 Distribution by function

A probability distribution can also be given by a function.

The domain of X must be specified, as the sum of the probabilities must equal 1.

Probability distribution by function

$P(X=x)=k\left(\frac{1}{3}\right)^{x-1}$ for $x=1,2,3$. Find constant k.

1.	Use the fact that $\sum P(X=x)=1$	$k\left(\frac{1}{3}\right)^{1-1}+k\left(\frac{1}{3}\right)^{2-1}+k\left(\frac{1}{3}\right)^{3-1}=1$
2.	Simplify and solve for k	$k+\frac{1}{3}k+\frac{1}{9}k=\frac{13}{9}k=1$. So, $k=\frac{9}{13}$.

7.3.2 Binomial distribution

Binomial distribution $X \sim B(n,p)$ used to find probabilities in situations with only 2 possible outcomes and lots of trials

$$P(X=x)=\binom{n}{r}p^{r}\left(1-p\right)^{n-r}$$

where $\binom{n}{r}=\mathrm{nCr}=\frac{n!}{r!(n-r)!}$
$n=$ number of trials
$p=$ probability of success
$r=$ number of success

You can calculate values using binomial expansion from the algebra chapter. However binomial distribution questions are often found on calculator papers.

For questions asking for the probability of an exact outcome, $P(X=r)$, we use `Binompdf` on the GDC.

For questions asking for the probability of several consecutive values, $P(X \leq r)$, we use `Binomcdf` on the GDC.

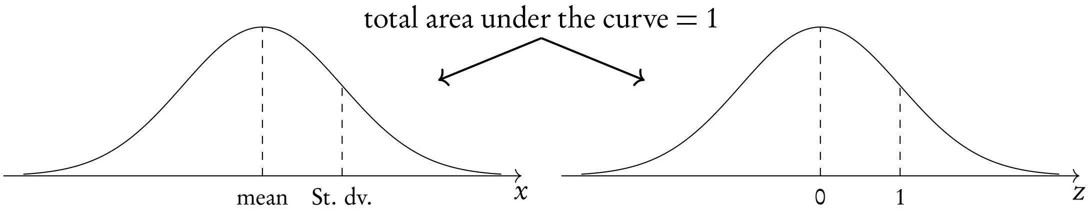
Note that `Binomcdf` only calculates $P(X \leq r)$ or in words "at most the value of r". Therefore you must remember to transform the function depending on the wording in the questions:

- "Less than r" $P(X<r)=P(X \leq r-1)$
- "More than r" $P(X>r)=1-P(X \leq r)$
- "At least r" $P(X \geq r)=1-P(X \leq r-1)$

7.3.3 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted.

We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two x-values always corresponds to the probability for getting an x-value in this range. The total area under the normal distribution is always 1; this is because the total probability of getting any x-value adds up to 1 (or, in other words, you are 100% certain that your x-value will lie somewhere on the x-axis below the bell-curve).



Even though you will be using your GDC to find probabilities for normal distributions, it's always *very* useful to draw a diagram to indicate for yourself (and the examiner) what area or x-value you are looking for.

Notation: $X \sim N(\mu, \sigma^2)$

Transform to standard N: $Z = \dfrac{x-\mu}{\sigma}$

On calculator:

`normcdf` (lower bound, upper bound, mean ($= \mu$), standard deviation ($= \sigma$))

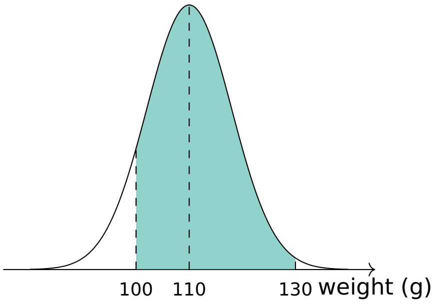
`invnorm` (area, mean ($= \mu$), standard deviation ($= \sigma$))

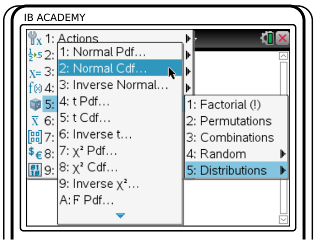
Finding a probability or percentage using normal distribution

The weights of pears are normally distributed with mean = 110 g and standard deviation = 8 g.
Find the percentage of pears that weigh between 100 g and 130 g

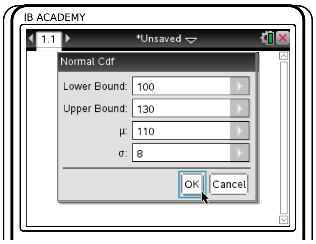
Sketch!
Indicate:

- The mean = 110 g
- Lower bound = 100 g
- Upper bound = 130 g
- And shade the area you are looking to find.



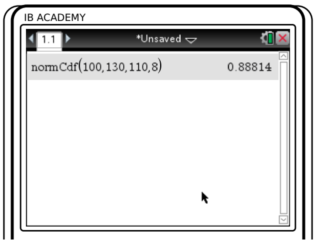


Press `menu`, choose
`5: Probability`
`5: Distributions`
`2: Normal Cdf`



Normal Cdf
Lower Bound: 100
Upper Bound: 130
μ: 110
σ: 8

Enter lower and upper boundaries, mean (μ) and standard deviation (σ).
For lower bound = $-\infty$,
set `lower: -1E99`
For upper bound = ∞,
set `upper: 1E99`



```
normCdf(100,130,110,8)        0.88814
```

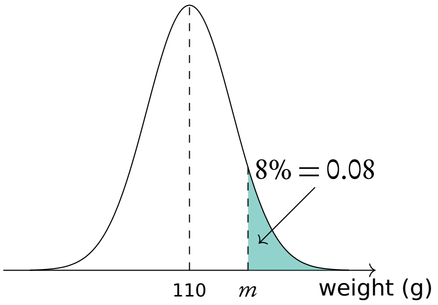
Press OK

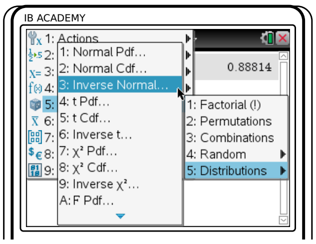
So 88.8% of the pears weigh between 100 g and 130 g.

Finding an x-value using normal distribution when the probability is given

The weights of pears are normally distributed with mean $= 110$ g and standard deviation $= 8$ g. 8% of the pears weigh more than m grams. Find m.

Sketch!



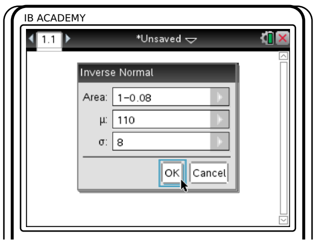


Press menu

`5: Probability`

`5: Distributions`

`3: Inverse Normal`



Area: 1−0.08

μ: 110

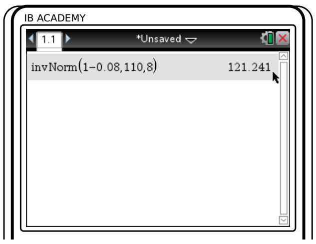
σ: 8

Enter probability (`Area`), mean (μ) and standard deviation (σ).

The calculator assumes the area is to the left of the x-value you are looking for.

So in this case:

$$\text{area} = 1 - 0.08 = 0.92$$



invNorm(1−0.08,110,8) 121.241

Press OK

So $m = 121$, which means that 8% of the pears weigh more than 121 g.

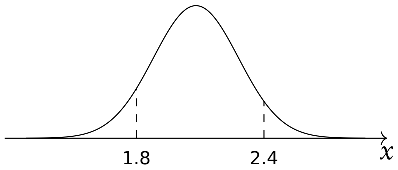
Finding mean and standard deviation of a normal distribution

All nails longer than 2.4 cm (5.5%) and shorter than 1.8 cm (8%) are rejected. What is the mean and standard deviation length?

1. Write down equations

$$P(L < 1.8) = 0.08$$
$$P(L > 2.4) = 0.055$$

2. Draw a sketch!



3. Write standardized equation of the form $P(Z < \ldots)$

$$P\left(Z < \frac{1.8-\mu}{\sigma}\right) = 0.08$$
$$P\left(Z > \frac{2.4-\mu}{\sigma}\right) = 0.055$$
$$P\left(Z < \frac{2.4-\mu}{\sigma}\right) = 1 - 0.055 = 0.945$$

4. Use `invnorm` on calculator

$$\texttt{invnorm}\,(0.08, 0, 1) = -1.4051$$
$$\texttt{invnorm}\,(0.945, 0, 1) = 1.5982$$

5. Equate and solve

Here you are solving a pair of simultaneous equations. For a review see the Prior Knowledge section.

$$\begin{cases} \dfrac{1.8-\mu}{\sigma} = -1.4051 \\ \dfrac{2.4-\mu}{\sigma} = 1.5982 \end{cases}$$
$$\begin{cases} \mu = 2.08 \\ \sigma = 0.200 \end{cases}$$

7.3.4 Continuous distribution

A more general type of distribution is known as a continuous distribution. It can be described by almost any function (usually piece-wise). You might be required to find mean or variance of such distributions, which can be done using integrals with formulas below.

Expected value $\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$

Variance $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$

DB 4.14

Since the function $f(x)$ is piece-wise, its value is 0 towards $\pm\infty$. Thus you do not have to take limits to infinity but only consider where the function is well defined.

It is also important to remember that the total probability of the distribution is always 1.

Example.

Find the value a, mean and variance of the following continuous distribution:

$$f(x) = \begin{cases} 2x & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \int_0^a 2x\,\mathrm{d}x &= 1 \\ a^2 - 0^2 &= 1 \\ a &= 1 \end{aligned}$$

$$\begin{aligned} \mathrm{E}(X) &= \int x f(x)\,\mathrm{d}x \\ &= \int_0^1 x2x\,\mathrm{d}x \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \int x^2 f(x)\,\mathrm{d}x - \mu^2 \\ &= \int_0^1 x^2 2x\,\mathrm{d}x - \frac{4}{9} \\ &= 0.5 - \frac{4}{9} \\ &= \frac{1}{18} \end{aligned}$$

7.4 Bayes theorem

Bayes theorem

1. the probability that B is true given that A is true

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B')P(A|B')}$$

2. the probability that one of the events B_i is true given that A is true, for $i = 1, 2, 3$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

Bayes theorem is very similar to conditional probability, since it is derived from that formula. The first formula should be used when there are two possible outcomes, while the second should be used when there are three possible events or outcomes.

Bayes theorem is applicable when we try to find probability of event B being true, given that event A is true.

STATISTICS

8

Table of contents & cheatsheet

Definitions

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).

Sample the observations actually selected from the population for a statistical test.

Random Sample a sample that is selected from the population with no bias or criteria; the observations are made at random.

Discrete finite or countable number of possible values (e.g. money, number of people)

Continuous infinite amount of increments (e.g. time, weight)

Note: continuous data can be presented as discrete data, e.g. if you round time to the nearest minute or weight to the nearest kilogram.

8.1. Descriptive statistics 130

For 1 variable data with frequency use `1-Var Stats` on GDC.

Mean the average value

$$\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}}$$

Mode the value that occurs most often

Median when the data set is ordered low to high and the number of data points is:

- odd, then the median is the middle value;
- even, then the median is the average of the two middle values.

Range largest x-value − smallest x-value

Variance $\sigma^2 = \dfrac{\sum f(x-\bar{x})^2}{n}$ *calculator only*

Standard deviation $\sigma = \sqrt{\text{variance}}$ *calculator only*

Grouped data data presented as an interval

Use the midpoint as the x-value in all calculations.

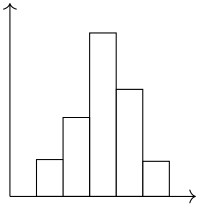
Q_1	first quartile = 25th percentile
Q_2	median = 50th percentile
Q_3	third quartile = 75th percentile
$Q_3 - Q_1$	interquartile range (IQR) = middle 50 percent

8.3. Statistical graphs 133

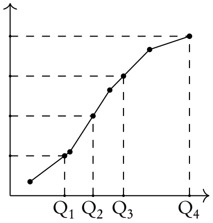
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

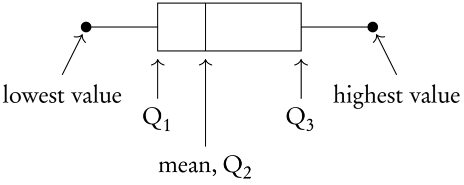
Histogram



Cumulative frequency



Box and whisker plot



8.4. Bivariate statistics 137

For analysis of data with two variables.
On GDC use `LinReg(ax+b)`.

Regression Line ($r = ax + b$)

Can be used to interpolate unknown data.

Interpretation of r-values

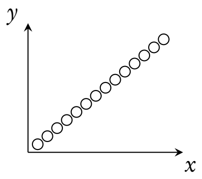
The correlation between the two sets of data. Can be positive or negative.

r-value	correlation
$0.00 \leq \lvert r \rvert \leq 0.25$	very weak
$0.25 \leq \lvert r \rvert \leq 0.50$	weak
$0.50 \leq \lvert r \rvert \leq 0.75$	moderate
$0.75 \leq \lvert r \rvert \leq 1.00$	strong

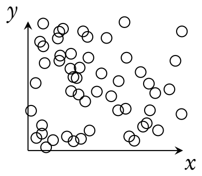
Correlation does not mean causation.

Scatter diagrams

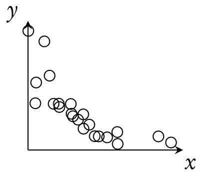
Perfect positive



No correlation



Weak negative



8.1 Descriptive statistics

The mean, mode and median, are all ways of measuring "averages". Depending on the distribution of the data, the values for the mean, mode and median can differ slightly or a lot. Therefore, the mean, mode and median are all useful for understanding your data set.

Example data set: 6, 3, 6, 13, 7, 7 in a table:

x	3	6	7	13
frequency	1	2	2	1

Mean the average value, $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}} = \dfrac{\sum x}{n} = \dfrac{\sum f x}{\sum f}$

Mode the value that occurs most often (highest frequency)

Median the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values. Find for larger values as $n + \frac{1}{2}$.

Range largest x-value − smallest x-value

Variance $\sigma^2 = \dfrac{\sum f(x-\bar{x})^2}{n}$ *calculator only*

Standard deviation $\sigma = \sqrt{\text{variance}}$ *calculator only*

Grouped data data presented as an interval, e.g. $10 < x \le 20$ where:

- lower boundary $= 10$
- upper boundary $= 20$
- interval width $= 20 - 10 = 10$
- mid-interval value (midpoint) $= \dfrac{20+10}{2} = 15$

Use the midpoint as the x-value in all calculations with grouped data.

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

Table 8.1: Adding or multiplying by a constant

	adding constant k	**multiplying by k**
mean	$\bar{x} + k$	$k \times \bar{x}$
standard deviation	σ	$k \times \sigma$

Q_1	the value for x so that 25% of all the data values are $\leq$ to it first quartile	= 25th percentile
Q_2	median	= 50th percentile
Q_3	third quartile	= 75th percentile
$Q_3 - Q_1$	interquartile range (IQR)	= middle 50 percent

Example.

Snow depth is measured in centimetres:
$30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110$.
Find the range, the median, the lower quartile, the upper quartile and the interquartile range.

First always rearrange data into ascending order: $30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125$

1. The range:
$$125 - 30 = 95\,\text{cm}$$
2. The median: there are 12 values so the median is between the 6th and 7th value.
$$\frac{65+70}{2} = 67.5\,\text{cm}$$
3. The lower quartile: there are 12 values so the lower quartile is between the 3rd and 4th value.
$$\frac{55+60}{2} = 57.5\,\text{cm}$$
4. The upper quartile: there are 12 values so the lower quartile is between the 9th and 10th value.
$$\frac{75+110}{2} = 92.5\,\text{cm}$$
5. The IQR
$$92.5 - 57.5 = 35\,\text{cm}$$

8.2 Sampling techniques

In order to do estimations on the whole population, it is required to create samples that can represent the population. There are different sampling techniques that achieve this goal, where all of them have different advantages, flaws and objectives. It is good to be able to know the difference between each one of them.

Convenience sampling the sampling done on the easiest available members of the population.

e.g. if you wanted to do a survey on students in your school, you would ask students you are personally familiar with.

Simple random sampling the sampling where each member of the population has an equal chance of being selected.

e.g. you could randomly put all students' names in a hat and randomly select sample members out of it.

Systematic sampling the sampling where the population is arranged or listed in a specific order and then elements from that list are selected at fixed intervals starting at a random point.

e.g. all student names could be written in an alphabetical list and every 10th student is chosen.

Stratified sampling the population is split into several smaller groups, known as "strata". Then a random sample is selected from each strata.

e.g. students could be split according to their age groups, so that randomly several students from each age group are taken into a sample.

Quota sampling similarly to stratified sampling the population is split into groups. However, sampling from each group is done in a non-random manner. E.g. sampling could be done in proportion to the size of each strata.

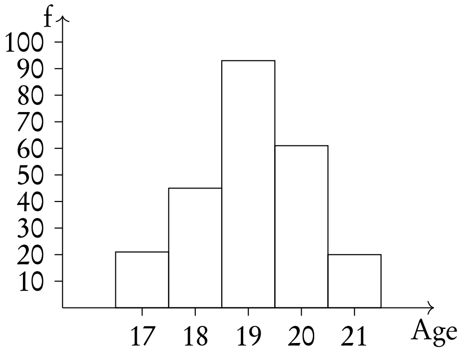
e.g. if the students are split into females and males with ratio 70% to 30%, then your sample should contain about 70% females and 30% males.

8.3 Statistical graphs

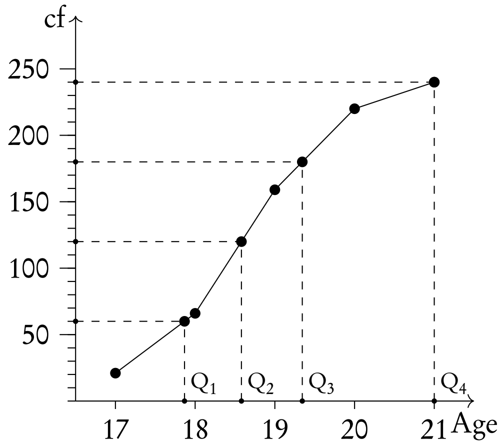
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

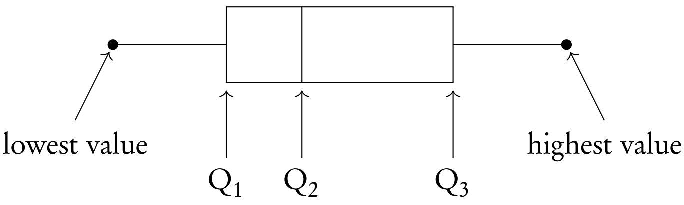
Age	17	18	19	20	21
No. of students	21	45	93	61	20
Cumulative freq.	21	66	159	220	240



A histogram is used to display the frequency for a specific condition. The frequencies (here: # of students) are displayed on the y-axis, and the different classes of the sample (here: age) are displayed on the x-axis. As such, the differences in frequency between the different classes assumed in the sample can easily be compared.



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the x-axis, and the frequency is displayed on the y-axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.



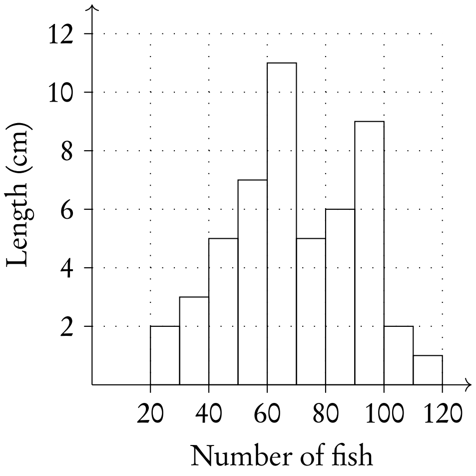
Box and whisker plots neatly summarize the distribution of the data. It gives information about the range, the median and the quartiles of the data. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line in the interior of the box, and the maximum and minimum points are at the ends of the whiskers.

Outliers will be any points lower than $Q_1 - 1.5 \times IQR$ and larger than $Q_3 + 1.5 \times IQR$ (IQR = interquartile range)

To identify the value of Q_1, Q_2 and Q_3, it is easiest to use the cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency (y-value) at which 25% for Q_1 / 50% for Q_2 / 75% for Q_3 of the sample is represented. To find the x-value, find the corresponding x-value for the previously identified y-value.

Example.

Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.

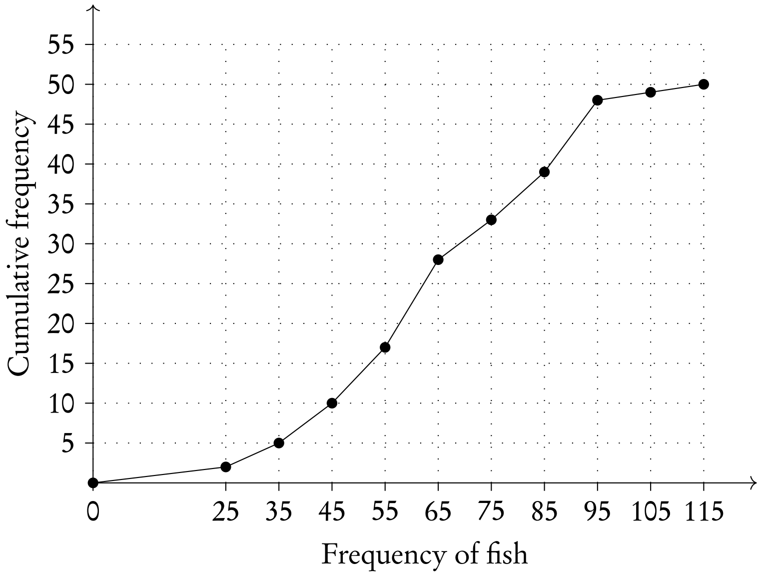


Write out the table for frequency and cumulative frequency.

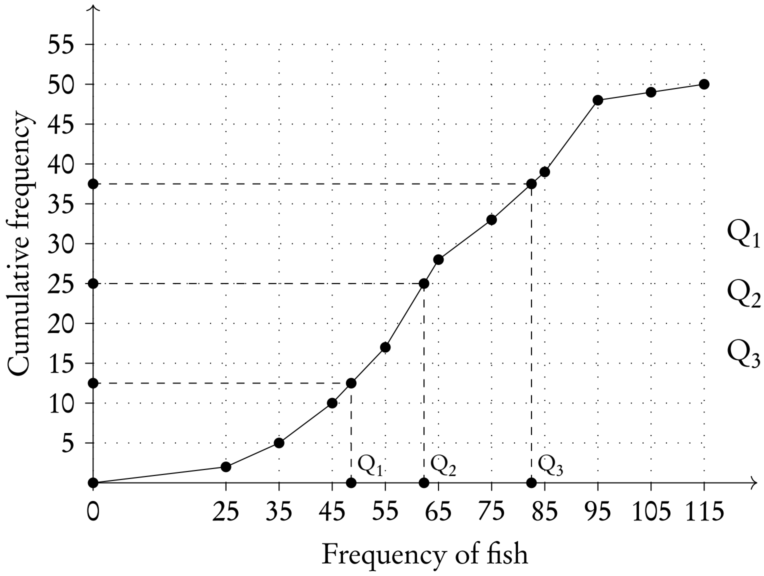
Frequency of fish	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Length of fish	2	3	5	7	11	5	6	9	1	1
Cumulative f.	2	5	10	17	28	33	39	48	49	50

Example.

Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find Q_1, Q_2 and Q_3.

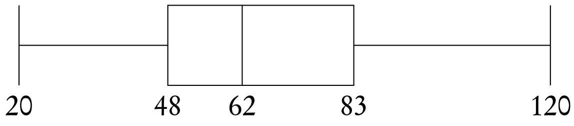


Q_1 25% of 50 $= 12.5$ $\rightarrow 48$

Q_2 50% of 50 $= 25$ $\rightarrow 62$

Q_3 75% of 50 $= 37.5$ $\rightarrow 83$

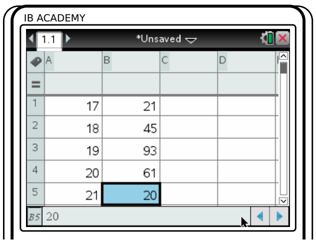
Plot box and whiskers.



GDC

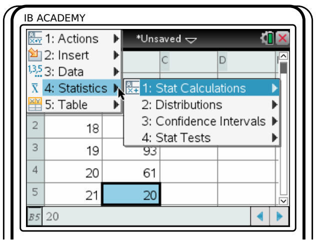
Finding the mean, standard deviation and quartiles etc.

Find the descriptive statistics for the data used in the previous example, showing the ages of students.

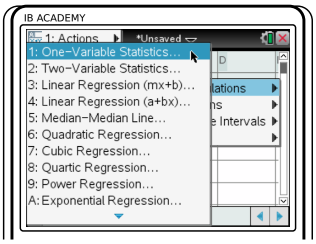


	A	B	C	D
1	17	21		
2	18	45		
3	19	93		
4	20	61		
5	21	20		

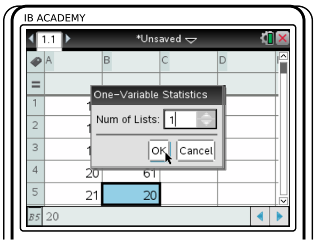
Press off on, go to `Lists and Spreadsheets`. Enter x-values in L1 and, if applicable, frequencies in L2



Press menu, choose `4: Statistics` `1: Stat Calculations`



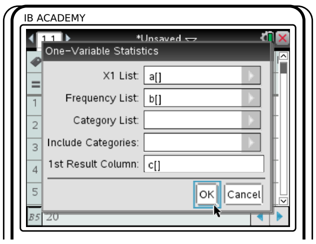
`1: One-Variable Statistics`



Num of Lists: 1

Enter `Num of lists: 1`.

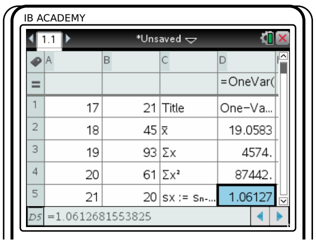
Press OK



X1 List: a[]
Frequency List: b[]
Category List:
Include Categories:
1st Result Column: c[]

Enter names of columns you used to enter your x-list and frequency list and column where you would like the solutions to appear: `a[]`, `b[]` and `c[]`.

Press OK



	A	B	C	D
=				=OneVar(
1	17	21	Title	One–Va…
2	18	45	$\bar{x}$	19.0583
3	19	93	Σx	4574.
4	20	61	Σx²	87442.
5	21	20	sx := sn-…	1.06127

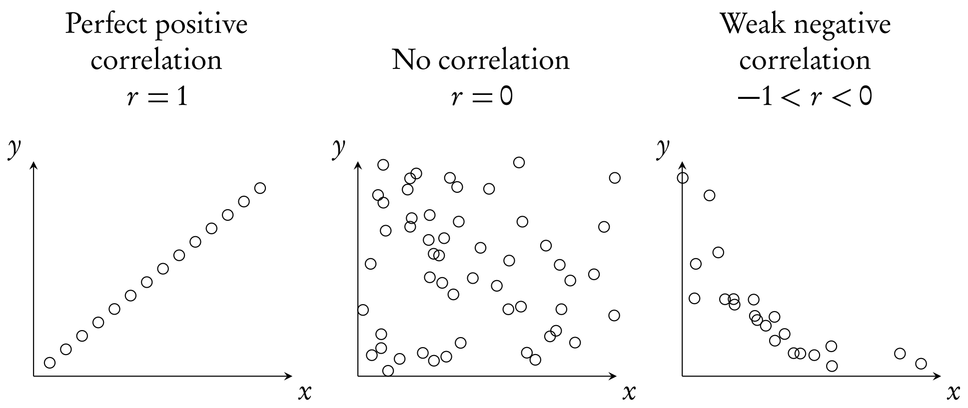
D5 =1.0612681553825

mean $= 19.06$; standard deviation $= 1.06$ etc.

8.4 Bivariate statistics

Bivariate statistics are about relationships between two different variables. You can plot your individual pairs of measurements as (x, y) coordinates on a scatter diagram. Analysing bivariate data allows you to assess the relationship between the two measured variables; we describe this relationship as correlation.

Scatter diagrams



Perfect positive correlation $r = 1$

No correlation $r = 0$

Weak negative correlation $-1 < r < 0$

Through statistical methods, we can predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. In your exam you will be expected to find linear regression models using your GDC.

8.4.1 Regression line

The regression line is a linear mathematical model describing the relationship between the two measured variables. This can be used to find an estimated value for points for which we do not have actual data. It is possible to have two different types of regression lines: y on x (equation $y = ax + b$), which can estimate y given value x, and x on y (equation $x = yc + d$), which can estimate x given value y. If the correlation between the data is perfect, then the two regression lines will be the same.

However one has to be careful when extrapolating (going further than the actual data points) as it is open to greater uncertainty. In general, it is safe to say that you should not use your regression line to estimate values outside the range of the data set you based it on.

8.4.2 Pearson's correlation coefficient ($-1 \leq r \leq 1$)

Besides simply estimating the correlation between two variables from a scatter diagram, you can calculate a value that will describe it in a standardised way. This value is referred to as Pearson's correlation coefficient (r).

$r = 0$ means no correlation.
$r \pm 1$ means a perfect positive/negative correlation.

Interpretation of r-values:

r—**value**	$0 < \lvert r \rvert \leq 0.25$	$0.25 < \lvert r \rvert \leq 0.50$	$0.50 < \lvert r \rvert \leq 0.75$	$0.75 < \lvert r \rvert < 1$
correlation	very weak	weak	moderate	strong

Remember that correlation $\neq$ causation.

Calculate r while finding the regression equation on your GDC. Make sure that `STAT DIAGNOSTICS` is turned `ON` (can be found in the `MODE` settings), otherwise the r-value will not appear.

When asked to "comment on" an r-value make sure to include both, whether the correlation is:

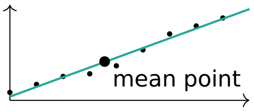
1. positive / negative
   and
2. strong / moderate / weak / very weak

Bivariate-statistics type questions

The height of a plant was measured the first 8 weeks

Week x	0	1	2	3	4	5	6	7	8
Height (cm) y	23.5	25	26.5	27	28.5	31.5	34.5	36	37.5

1. Plot a scatter diagram



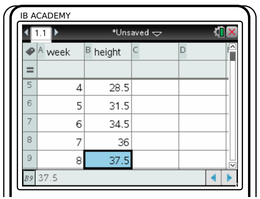
2. Use the mean point to draw a best fit line

$$\bar{x} = \frac{0+1+2+\ldots+8}{9} = 3.56$$

$$\bar{y} = \frac{23.5+25+\ldots+37.5}{9} = 30$$

The line of best fit should pass through the mean point.

3. Find the equation of the regression line Using GDC $y = 1.83x + 22.7$



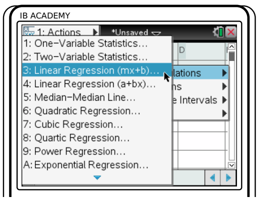
	A week	B height	C	D
=				
5	4	28.5		
6	5	31.5		
7	6	34.5		
8	7	36		
9	8	37.5		

B9 37.5

off

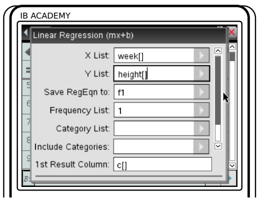
Press on, got to “Lists and Spreadsheets”

Enter x-values in one column (e.g A) and y-values in another column (e.g. B)



1: One−Variable Statistics…
2: Two−Variable Statistics…
3: Linear Regression (mx+b)…
4: Linear Regression (a+bx)…
5: Median−Median Line…
6: Quadratic Regression…
7: Cubic Regression…
8: Quartic Regression…
9: Power Regression…
A: Exponential Regression…

Press menu
4: Statistics
1: Stat Calculations
3: Linear Regression (mx+b)

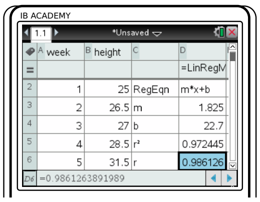


Linear Regression (mx+b)

X List: week[]
Y List: height[]
Save RegEqn to: f1
Frequency List: 1
Category List:
Include Categories:
1st Result Column: c[]

Enter
X list: A [];
Y list: B[];
1st Result Column: C[]

Press OK



	A week	B height	C	D
=				=LinRegM
2	1	25	RegEqn	m*x+b
3	2	26.5	m	1.825
4	3	27	b	22.7
5	4	28.5	r²	0.972445
6	5	31.5	r	0.986126

D6 =0.9861263891989

So, equation of regression line is $y = 1.83x + 22.7$ and Pearson’s correlation (r-value) $= 0.986$

4. Comment on the result.

Pearson’s correlation is $r = 0.986$, which is a strong positive correlation.